An Investigation of the Casimir Energy for a Fermion Coupled to the Sine-Gordon Soliton with Parity Decomposition

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Outline

1. Introduction

- *Soliton*
- *Casimir energy*

2. Casimir energy in coupled fermion-soliton systems

- *Direct method*
- *Indirect method*

3. Summary & Conclusion

Soliton :
- Solitary wave :
  - ⊛ has localized energy density.
  - ⊛ has finite energy.
  - ⊛ maintains its shape when it moves at constant speed.
- ⊛ emerges from the "collision" unchanged, except possibly for a phase shift.

Solitons in $1+1$ dimensions:

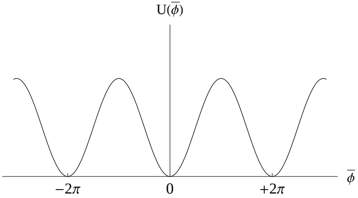
$$\mathcal{L}(x,t) = \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\phi')^2 - U(\phi)$$

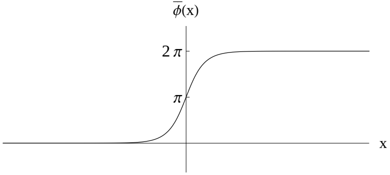
sine-Gordon theory:

$$U(\phi) = \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) - 1\right]$$

change of variables: $\bar{x} = mx, \quad \bar{t} = mt, \quad \bar{\phi} = (\sqrt{\lambda}/m)\phi$

$$\bar{\phi}(\bar{x}) = \pm 4\tan^{-1}[\exp(\bar{x} - \bar{x}_0)]$$



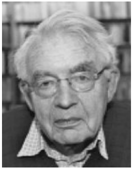


Topological charge:

$$Q \propto [\phi(x = \infty) - \phi(x = -\infty)]$$

Topological solitons: $Q \neq 0$
Non-topological solitons: $Q = 0$

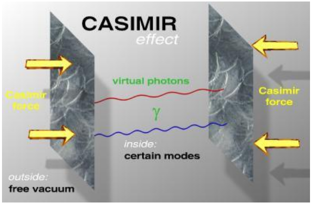
The Casimir energy and Casimir-Polder force arise from second quantization in the quantum field theory.



Consider two uncharged metallic plates that are parallel. They were set in the vacuum without any external electromagnetic field, placed a few micrometers apart (the distance between the objects should be extremely small!).

In the classical description, the lack of an external field means that there is no field between the plates, and no force would be measured between them.

This force arises from quantum fluctuation of the vacuum!



Quantum field theory treats each mode of electromagnetic field as a quantum harmonic oscillator. Thus the zero point energy of each mode is $\hbar\omega/2$. Summing over all possible modes gives an infinite quantity!

Lagrangian density for the coupled fermion-soliton system in (1+1) dimensions:

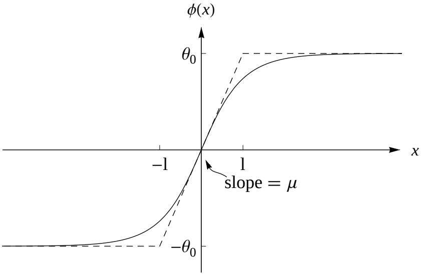
$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m e^{i\phi(x)\gamma^{5}}\right)\psi$$

$$\phi(x) = \begin{cases} -\theta_0 & \text{for} \quad x \leqslant -l \\ \mu x & \text{for} \quad -l \leqslant x \leqslant l \\ +\theta_0 & \text{for} \quad l \leqslant x \end{cases} \qquad \text{SESM}$$

$$\phi(x) = (m'/\sqrt{\lambda})\left\{\tan^{-1}\left[\exp(m'x)\right] - \pi\right\} \qquad \text{sine-Gordon}$$

m: fermion field mass

m': pseudoscalar field mass



dashed line: SESM
solid line: sine-Gordon

$$\psi(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \left[b_k u_k(x,t) + d_k^\dagger v_k(x,t) \right]$$
$$= \int_{0}^{+\infty} \frac{dp}{2\pi} \sum_{j=\pm} \left[a_p^j \mu_p^j(x,t) + c_p^{j\dagger} \nu_p^j(x,t) \right]$$
$$+ \sum_i \left[e_i \chi_{1b_i}(x,t) + f_i^\dagger \chi_{2b_i}(x,t) \right]$$

$$H = \int_{-\infty}^{+\infty} dx \left[\psi^\dagger(x,t) \mathcal{H} \psi(x,t) \right]$$

Casimir energy in coupled fermion-soliton systems

$$H = \int_{-\infty}^{+\infty} dx \Bigg\{ \int_0^{+\infty} \frac{dp}{2\pi} \sum_{j=\pm} \left[a_p^{j\dagger} \mu_p^{j\dagger}(x,t) + c_p^j \nu_p^{j\dagger}(x,t) \right]$$

$$+ \sum_i \left[e_i^\dagger \chi_{1b_i}^\dagger(x,t) + f_i \chi_{2b_i}^\dagger(x,t) \right] \Bigg\}$$

$$\times \Bigg\{ \int_0^{+\infty} \frac{dq}{2\pi} \sum_{n=\pm} \left[\sqrt{q^2+m^2} a_q^n \mu_q^n(x,t) - \sqrt{q^2+m^2} c_q^{n\dagger} \nu_q^n(x,t) \right]$$

$$+ \sum_l \left[E_{\text{bound}}^{l,\text{sky}} e_l \chi_{1b_l}(x,t) + E_{\text{bound}}^{l,\text{sea}} f_l^\dagger \chi_{2b_l}(x,t) \right] \Bigg\}$$

$$\langle\Omega\,|H|\,\Omega\rangle = \int_{-\infty}^{+\infty} dx \int_{0}^{+\infty} \frac{dp}{2\pi} \sum_{j=\pm} \left(-\sqrt{p^2+m^2}\right) \nu_p^{j\dagger}(x,t)\nu_p^{j}(x,t) + \int_{-\infty}^{+\infty} dx \sum_{i} \left(E_{\text{bound}}^{i,\text{sea}}\right) \chi_{2b_i}^{\dagger}(x,t)\chi_{2b_i}(x,t)$$

$$\begin{aligned} E_{\text{Casimir}} &= \langle\Omega\,|H|\,\Omega\rangle - \langle 0\,|H_{\text{free}}|\,0\rangle \\ &= \int_{-\infty}^{+\infty} dx \int_{0}^{+\infty} \frac{dp}{2\pi} \sum_{j=\pm} \left(-\sqrt{p^2+m^2}\right) \nu_p^{j\dagger}\nu_p^{j} \\ &+ \int_{-\infty}^{+\infty} dx \sum_{i} \left(E_{\text{bound}}^{i,\text{sea}}\right) \chi_{2b_i}^{\dagger}\chi_{2b_i} \\ &- \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \left(-\sqrt{k^2+m^2}\right) v_k^{\dagger}v_k \end{aligned}$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

representation: $\gamma^0 = \sigma_1,\ \gamma^1 = i\sigma_3,\ \gamma^5 = \gamma^0\gamma^1 = \sigma_2$

$$\xi = e^{-iEt}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 + i\psi_2 \\ \psi_1 - i\psi_2 \end{pmatrix}$$

$$\begin{pmatrix} i\partial_x - E & im\mathrm{e}^{i\phi(x)} \\ -im\mathrm{e}^{-i\phi(x)} & -i\partial_x - E \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(\partial_x^2 \mp i\theta'\partial_x + E^2 - m^2 + \theta' E)\xi_{1,2} = 0$$

SESM:

$$\xi_b(x) = \begin{cases} b\begin{pmatrix} 1 \\ c_- \end{pmatrix} e^{\lambda(x+l)}, & c_- = -\frac{\lambda+iE}{m}e^{i\theta_0}, \quad x \leqslant -l \\ \begin{pmatrix} de^{ik_1 x} + ee^{ik_2 x} \\ df_+ e^{-ik_2 x} + ef_- e^{-ik_1 x} \end{pmatrix}, & f_\pm = -i\frac{k_{1,2}+E}{m}, \quad -l \leqslant x \leqslant l \\ a\begin{pmatrix} 1 \\ c_+ \end{pmatrix} e^{-\lambda(x-l)}, & c_+ = \frac{\lambda-iE}{m}e^{-i\theta_0}, \quad l \leqslant x \end{cases}$$

$$a \equiv N(\text{Normalization})$$

$$b = Ne^{-i\theta_0}\left[\frac{i\mu + 2\lambda}{\zeta}\sin(\zeta l) + \cos(\zeta l)\right]$$

$$d = -Ne^{-ik_1 l}\frac{k_2 - i\lambda}{\zeta}$$

$$e = Ne^{-ik_2 l}\frac{k_1 - i\lambda}{\zeta}$$

$$N = \left\{ \frac{2}{\lambda} + \frac{4}{[\mathrm{Re}\,(\zeta)]^2} \left[\frac{\mu E - 2\lambda^2}{\mathrm{Re}\,(\zeta)} \sin\left[2l\mathrm{Re}\,(\zeta)\right] - \lambda \cos\left[2l\mathrm{Re}\,(\zeta)\right] + \lambda + l\mu\,(\mu + 2E) \right] \right\}^{-1/2}$$

$$\lambda \equiv \sqrt{M^2 - E^2}$$

$$\zeta \equiv \sqrt{\mu^2 - 4(\lambda^2 - \mu E)}$$

$$k_{1,2} \equiv \frac{1}{2}(\mu \pm \zeta)$$

Continuum scattering states:

$$\xi_0(x) = \begin{cases} a_0 \begin{pmatrix} im\mathrm{e}^{-i\theta_0/2} \\ (E+p)\mathrm{e}^{i\theta_0/2} \end{pmatrix} \mathrm{e}^{ip(x+l)} + b_0 \begin{pmatrix} im\mathrm{e}^{-i\theta_0/2} \\ (E-p)\mathrm{e}^{i\theta_0/2} \end{pmatrix} \mathrm{e}^{-ip(x+l)}, & x \leqslant -l \\ e_0 \begin{pmatrix} (k_1+E)\mathrm{e}^{i\mu x/2} \\ -im\mathrm{e}^{-i\mu x/2} \end{pmatrix} \mathrm{e}^{-i\xi x/2} + f_0 \begin{pmatrix} im\mathrm{e}^{i\mu x/2} \\ (k_1+E)\mathrm{e}^{-i\mu x/2} \end{pmatrix} \mathrm{e}^{i\xi x/2}, & -l \leqslant x \leqslant l \\ c_0 \begin{pmatrix} (E-p)\mathrm{e}^{i\theta_0/2} \\ -im\mathrm{e}^{-i\theta_0/2} \end{pmatrix} \mathrm{e}^{ip(x-l)}, \quad l \leqslant x \end{cases}$$

$$a_0 = \frac{-i}{2mp}[-(k_1-p)^2(E-p)^2\mathrm{e}^{i\xi l} + (k_1+p)^2 m^2 \mathrm{e}^{-i\xi l}]$$

$$b_0 = \frac{m(k_1^2-p^2)}{p}\sin(\xi l), \qquad e_0 = (k_1-p)(E-p)\mathrm{e}^{(i/2)\xi l}$$

$$c_0 = (k_1+E)^2 - m^2, \qquad f_0 = -im(k_1+p)\mathrm{e}^{-(i/2)\xi l}$$

Continuum parity eigenstates:

$$\xi_\pm(x) = \begin{cases} a_\pm \begin{pmatrix} im\mathrm{e}^{-i\theta_0/2} \\ (E+p)\mathrm{e}^{i\theta_0/2} \end{pmatrix} \mathrm{e}^{ip(x+l)} + b_\pm \begin{pmatrix} im\mathrm{e}^{-i\theta_0/2} \\ (E-p)\mathrm{e}^{i\theta_0/2} \end{pmatrix} \mathrm{e}^{-ip(x+l)}, & x \leqslant -l \\ ic_\pm \begin{pmatrix} (k_1+E)\mathrm{e}^{i\mu x/2} \\ -im\mathrm{e}^{-i\mu x/2} \end{pmatrix} \mathrm{e}^{-i\xi x/2} \pm c_\pm \begin{pmatrix} im\mathrm{e}^{i\mu x/2} \\ (k_1+E)\mathrm{e}^{-i\mu x/2} \end{pmatrix} \mathrm{e}^{i\xi x/2}, & -l \leqslant x \leqslant l \\ \pm ia_\pm \begin{pmatrix} (E+p)\mathrm{e}^{i\theta_0/2} \\ -im\mathrm{e}^{-i\theta_0/2} \end{pmatrix} \mathrm{e}^{-ip(x-l)} \pm ib_\pm \begin{pmatrix} (E-p)\mathrm{e}^{i\theta_0/2} \\ -im\mathrm{e}^{-i\theta_0/2} \end{pmatrix} \mathrm{e}^{ip(x-l)}, & l \leqslant x \end{cases}$$

$$a_\pm = \frac{c_\pm}{2pm}\left\{[m\mathrm{e}^{i\xi l/2} \pm (k_1+E)\mathrm{e}^{-i\xi l/2}]m - [(k_1+E)\mathrm{e}^{i\xi l/2} \pm m\mathrm{e}^{-i\xi l/2}](E-p)\right\}$$

$$b_\pm = -\frac{c_\pm}{2pm}\{[m\mathrm{e}^{i\xi l/2} \pm (k_1+E)\mathrm{e}^{-i\xi l/2}]m - [(k_1+E)\mathrm{e}^{i\xi l/2} \pm m\mathrm{e}^{-i\xi l/2}](E+p)\}$$

$$c_\pm \equiv \left\{ \cosh[\mathrm{Im}(\xi)\mathrm{l}][(\frac{|k_1^2|}{p^2}+1)2E^2 + 4E\mathrm{Re}(\mathrm{k}_1)] \mp \cos[\mathrm{Re}(\xi)\mathrm{l}]2mE(\frac{|k_1^2|}{p^2}-1)\right\}^{-1/2}$$

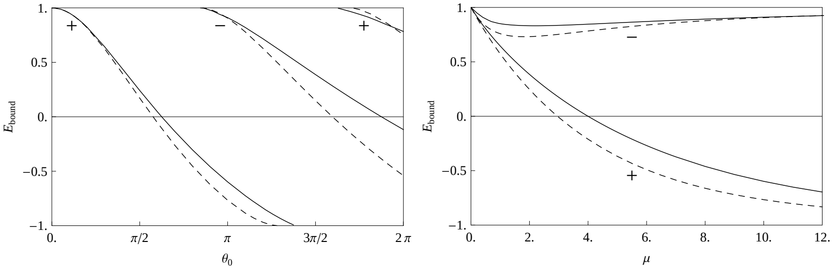
$$\phi(x) = (m'/\sqrt{\lambda})\left\{\tan^{-1}\left[\exp(m'x)\right] - \pi\right\} \qquad \text{sine-Gordon}$$

$$\xi_0(x) = \begin{pmatrix} \xi_1(x) \\ \xi_2(x) \end{pmatrix} = \begin{pmatrix} \eta_1(x) + i\eta_2(x) \\ \eta_3(x) + i\eta_4(x) \end{pmatrix}$$

The equations of motion:

$$\eta_1^{'} + m\cos\phi\,(x)\,\eta_3 - E\eta_2 - m\sin\phi\,(x)\,\eta_4 = 0$$
$$\eta_2^{'} + m\cos\phi\,(x)\,\eta_4 + E\eta_1 + m\sin\phi\,(x)\,\eta_3 = 0$$
$$\eta_3^{'} + m\cos\phi\,(x)\,\eta_1 + E\eta_4 + m\sin\phi\,(x)\,\eta_2 = 0$$
$$\eta_4^{'} + m\cos\phi\,(x)\,\eta_2 - E\eta_3 - m\sin\phi\,(x)\,\eta_1 = 0$$

$\theta = \pi$: $\mu = 10$:



solid line: sine-Gordon

dashed line: SESM

Continuum scattering states:

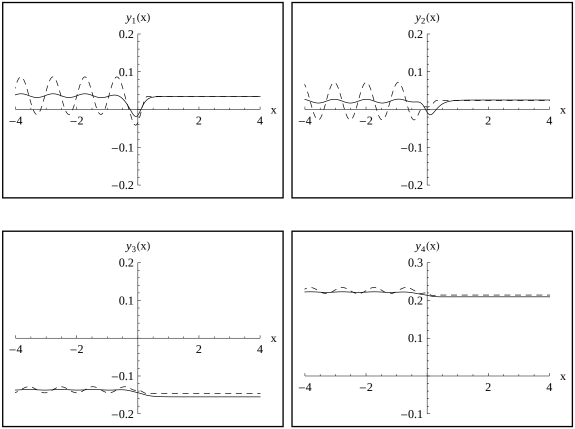
$$\xi_k(x) = \begin{pmatrix} y_1(x) + iy_2(x) \\ y_3(x) + iy_4(x) \end{pmatrix} = \begin{cases} \begin{pmatrix} a_1 + ia_2 \\ a_3 + ia_4 \end{pmatrix} \mathrm{e}^{-ikx} + \begin{pmatrix} b_1 + ib_2 \\ b_3 + ib_4 \end{pmatrix} \mathrm{e}^{ikx}, & x \to -\infty \\ \begin{pmatrix} z_1(x) + iz_2(x) \\ z_3(x) + iz_4(x) \end{pmatrix}, & \text{finite } x \\ \begin{pmatrix} c_1 + ic_2 \\ c_3 + ic_4 \end{pmatrix} \mathrm{e}^{ikx}, & x \to +\infty \end{cases}$$

$$\xi_0(x) = \mathrm{e}^{ikx} \begin{pmatrix} y_1(x) + iy_2(x) \\ y_3(x) + iy_4(x) \end{pmatrix}$$

The equations of motion satisfied by η_is:

$$\begin{aligned}
y_1^{'} + m\cos\phi\,(x)\,y_3 - (E+k)y_2 - m\sin\phi\,(x)\,y_4 &= 0 \\
y_2^{'} + m\cos\phi\,(x)\,y_4 + (E+k)y_1 + m\sin\phi\,(x)\,y_3 &= 0 \\
y_3^{'} + m\cos\phi\,(x)\,y_1 + (E-k)y_4 + m\sin\phi\,(x)\,y_2 &= 0 \\
y_4^{'} + m\cos\phi\,(x)\,y_2 - (E-k)y_3 - m\sin\phi\,(x)\,y_1 &= 0
\end{aligned}$$

$\theta_0 = \pi,\ \mu = 10,\ k = 3.0,\ E = +\sqrt{k^2 + M^2}$:



$$E_{\text{Casimir}} = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{0}^{+\infty} \frac{\mathrm{d}p}{2\pi} \sum_{j=\pm} \left(-\sqrt{p^2+m^2}\right) \nu_p^{j\dagger} \nu_p^j$$

$$+ \int_{-\infty}^{+\infty} \mathrm{d}x \sum_i \left(E_{\text{bound}}^{i-}\right) \chi_{2\mathrm{b}_i}^\dagger \chi_{2\mathrm{b}_i}$$

$$- \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \left(-\sqrt{k^2+m^2}\right) v_k^\dagger v_k$$

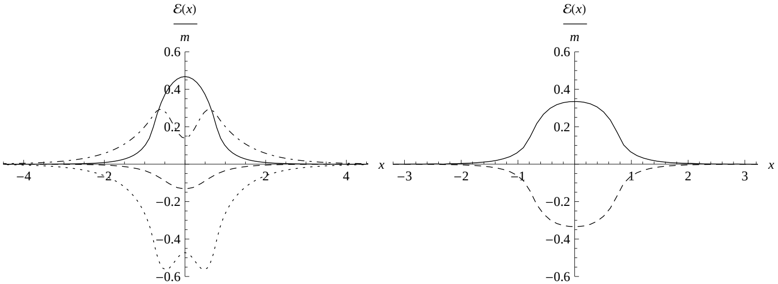
$$= \sum_i E_{\text{bound}}^{i-} - \int_0^{+\infty} \mathrm{d}k \sqrt{k^2+m^2} \left(\rho(k) - \rho_0(k)\right) + \frac{m}{2}$$

$$E_{\text{Casimir}} = \int_{-\infty}^{+\infty} \mathrm{d}x \varepsilon_{\text{sea}}(x) + \sum_i E_{\text{bound}}^{i-} = -\left(\int_{-\infty}^{+\infty} \mathrm{d}x \varepsilon_{\text{sky}}(x) + \sum_i E_{\text{bound}}^{i+}\right)$$

$$= \frac{1}{2}\left(\int_{-\infty}^{+\infty} \mathrm{d}x \varepsilon_{\text{sea}}(x) + \sum_i E_{\text{bound}}^{i-}\right) - \frac{1}{2}\left(\int_{-\infty}^{+\infty} \mathrm{d}x \varepsilon_{\text{sky}}(x) + \sum_i E_{\text{bound}}^{i+}\right)$$

Casimir energy in coupled fermion-soliton systems

SESM:



left graph:

solid (dotted) line: density of continuum states with negative (positive) energy
dashed (dotdashed) line: density of bound states with negative (positive) energy

right graph:

solid line: sum of the densities of states with negative energy
dashed line: sum of the densities of states with positive energy

$$\rho(k) - \rho_0(k) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}k}\delta_F(k)$$

$$\delta_F(k) = \delta^{\mathrm{sky}}(k) + \delta^{\mathrm{sea}}(k)$$

$$S(k) = \mathrm{e}^{i\delta_F(k)}$$

$$\delta^{\mathrm{sky}}(k) = \delta_+^{\mathrm{sky}}(k) + \delta_-^{\mathrm{sky}}(k)$$

$$\delta^{\mathrm{sea}}(k) = \delta_+^{\mathrm{sea}}(k) + \delta_-^{\mathrm{sea}}(k)$$

$$S_\pm(k) = \mathrm{e}^{2i\delta_\pm(k)}$$

$$y^{\pm}(x) = \xi(x) \pm \chi(x)$$

Parity condition:

$$Py^{\pm}(x) = -\sigma_2 y^{\pm}(x) = \pm y^{\pm}(-x)$$

Parity eigenstates:

$$y^{+}(x) = \begin{pmatrix} y_1^{+}(x) + iy_2^{+}(x) \\ y_3^{+}(x) + iy_4^{+}(x) \end{pmatrix}$$

$$y^{-}(x) = \begin{pmatrix} y_1^{-}(x) + iy_2^{-}(x) \\ y_3^{-}(x) + iy_4^{-}(x) \end{pmatrix}$$

Weak form of the Levinson theorem:

$$\begin{aligned}\Delta\delta &\equiv [\delta_{\rm sky}(0)-\delta_{\rm sky}(\infty)]+[\delta_{\rm sea}(0)-\delta_{\rm sea}(\infty)]\\ &=\left(N+\frac{N_{\rm t}}{2}-\frac{N_{\rm t}^0}{2}\right)\pi\end{aligned}$$

$$\begin{aligned}\Delta\delta_\pm &\equiv [\delta_\pm^{\rm sky}(0)-\delta_\pm^{\rm sky}(\infty)]+[\delta_\pm^{\rm sea}(0)-\delta_\pm^{\rm sea}(\infty)]\\ &=\left(N_\pm+\frac{N_{{\rm t},\pm}}{2}-\frac{N_{{\rm t},\pm}^0}{2}\right)\pi\end{aligned}$$

Strong form of the Levinson theorem:

$$\delta(0) = (N_{\text{exit}} - N_{\text{enter}})\,\pi$$

$$\delta_{\pm}(0) = (N_{\text{exit},\pm} - N_{\text{enter},\pm})\,\pi$$

$$\delta(\infty) = (N_{\text{enter}} - N_{\text{exit}})\,\pi$$

$$\delta_{\pm}(\infty) = (N_{\text{enter},\pm} - N_{\text{exit},\pm})\,\pi$$

$$\delta^{\text{sky}}_{\text{sea}}(\infty) = \pm\theta_0$$

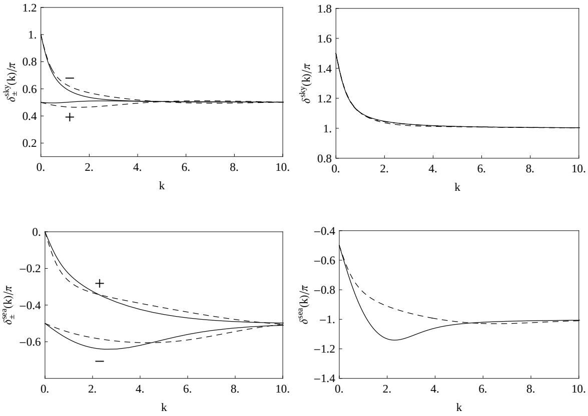
$$\delta^{\text{sky}}_{\pm}(\infty) = +\frac{\theta_0}{2}$$

$$\delta^{\text{sea}}_{\pm}(\infty) = -\frac{\theta_0}{2}$$

Casimir energy in coupled fermion-soliton systems

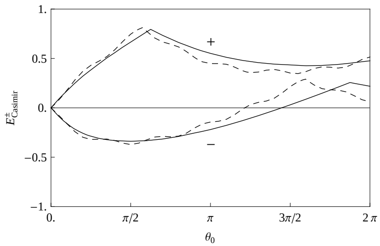
indirect method: the phase shift method

$\theta_0 = \pi$ and $\mu = 10$:



$N_{\text{sky}}^{\pm} = 1$ and $N_{t,\text{sky}}^{0,+}(N_{t,\text{sky}}^{0,-}) = 1(0)$ at $E = +m \rightarrow \delta_{+}^{\text{sky}}(0)(\delta_{-}^{\text{sky}}(0))/\pi = \frac{1}{2}(1)$

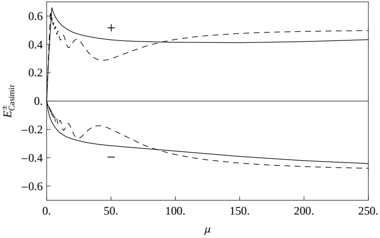
$\mu = 10$:

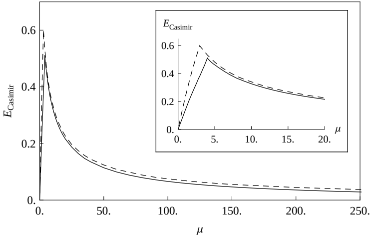


solid line: sine-Gordon

dashed line: SESM

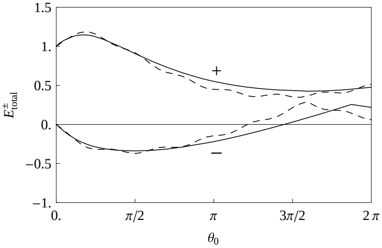
$\theta_0 = \pi$:

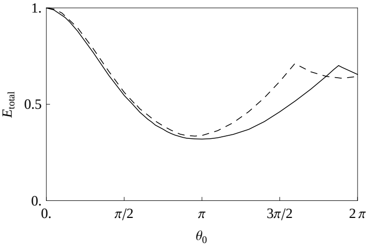




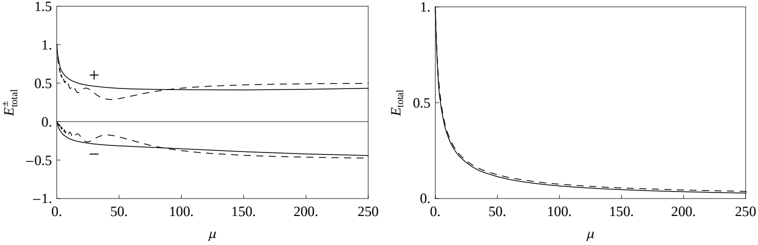
$$\mu \to \infty : \quad E_{\text{Casimir}} = \frac{m}{2} - \int_0^{+\infty} \frac{dp}{\pi} \frac{m^2 \sin(\theta_0)\cos(\theta_0)}{\left[p^2 + m^2 \sin^2(\theta_0)\right]} + \sum_i \left(E^{i,\text{sea}}_{\text{bound}} \right)$$

$\mu = 10$:





$\theta_0 = \pi$:


1.5
1.
0.5
0.
−0.5
−1.
+
−
$E^{\pm}_{\text{total}}$
0.
50.
100.
150.
200.
250
μ
1.
0.5
0.
E_{total}
0.
50.
100.
150.
200.
250
μ

Summary & Conclusion

- The Casimir energy in fermionic systems just originates from the sea, although in our models the positive and negative energy densities are mirror images of each other.
- The phase shift method is suitable to calculate the Casimir energy when the model is not exactly solvable.
- There is a maximum in the Casimir energy profile when a fermionic bound energy level crosses the line of $E = 0$. This shows that such a configuration, for all three models, are energetically unfavorable.
- The limits of the Casimir energy profile are consistent with our physical intuitions.

Summary & Conclusion

- Within the context of our models, the pseudoscalar fields are almost reflectionless barriers for the fermions.
- Considering the total energy of a system consisting of a valence fermion in the ground state, we conclude that the preference of the system is the background field with winding number one.

Thanks for your attention