Spectral Singularities, PT-Symmetry & Unidirectional Invisibility

Ali Mostafazadeh
Koç University, Istanbul

Outline:

- **Motivation: Pseudo-Hermitian QM**
- **Spectral Singularties**
    - As singularities of the metric operator
    - As zero-width resonances
    - **Self-dual spectral singularities & PT-symmetry**
- **Unidirectional Invisibility & PT-symmetry**

Pseudo-Hermitian Quantum Mechanics

Bender & Boettcher, PRL **80**, 5243 (1998):

$H = p^2 + ix^3$ has a real spectrum. Why?

Pseudo-Hermitian Quantum Mechanics

Bender & Boettcher, PRL **80**, 5243 (1998):

$H = p^2 + ix^3$ has a real spectrum. Why?

Ans 1: Because it commutes with $\mathcal{PT}$, where

$$\mathcal{P}\psi(x) = \psi(-x), \quad \mathcal{T}\psi(x) = \psi(x)^*$$

Pseudo-Hermitian Quantum Mechanics

Bender & Boettcher, PRL **80**, 5243 (1998):

$H = p^2 + ix^3$ has a real spectrum. Why?

Ans 1: Because it commutes with $\mathcal{PT}$, where

$$\mathcal{P}\psi(x) = \psi(-x), \quad \mathcal{T}\psi(x) = \psi(x)^*$$

This cannot be correct, because ix commutes with $\mathcal{PT}$ but its spectrum is imaginary.

Pseudo-Hermitian Quantum Mechanics

Bender & Boettcher, PRL **80**, 5243 (1998):

$H = p^2 + ix^3$ has a real spectrum. Why?

Ans 1: Because it commutes with $\mathcal{PT}$, where

$$\mathcal{P}\psi(x) = \psi(-x), \quad \mathcal{T}\psi(x) = \psi(x)^*$$

This cannot be correct, because ix commutes with $\mathcal{PT}$ but its spectrum is imaginary.

Ans 2: Because H has a complete set of eigenfunctions ψ_n that are also eigenfunctions of $\mathcal{PT}$.

Pseudo-Hermitian Quantum Mechanics

Bender & Boettcher, PRL **80**, 5243 (1998):

$H = p^2 + ix^3$ has a real spectrum. Why?

Ans 1: Because it commutes with $\mathcal{PT}$, where

$$\mathcal{P}\psi(x) = \psi(-x), \qquad \mathcal{T}\psi(x) = \psi(x)^*$$

This cannot be correct, because ix commutes with $\mathcal{PT}$ but its spectrum is imaginary.

Ans 2: Because H has a complete set of eigenfunctions ψ_n that are also eigenfunctions of $\mathcal{PT}$.

This is correct, but not easy to check. It also does not answer the qxn: "What if an operator does not share a complete set of eigenfunctions with $\mathcal{PT}$?"

E.g., $H = (p + \mathfrak{z}\, x)^2 + x^2$ or $H = p^2 + \mathfrak{z}\, \delta(x)$ with $\mathfrak{z} \in \mathbb{C}$.

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Let $H : \mathscr{H} \to \mathscr{H}$ be an operator with a discrete spectrum.

Diagonalizability of H means the existence of a complete biorthonormal eigensystem $\{(\phi_n, \psi_n)\}$:

$$H\psi_n = E_n\psi_n, \quad H^\dagger\phi_n = E_n^*\phi_n, \quad \langle\phi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Let $H : \mathscr{H} \to \mathscr{H}$ be an operator with a discrete spectrum.

Diagonalizability of H means the existence of a complete biorthonormal eigensystem $\{(\phi_n, \psi_n)\}$:

$$H\psi_n = E_n\psi_n, \quad H^\dagger\phi_n = E_n^*\phi_n, \quad \langle\phi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Qxn: What are the necessary and sufficient conditions for an operator to be diagonalizable and have a real spectrum.

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Let $H : \mathscr{H} \to \mathscr{H}$ be an operator with a discrete spectrum.

Diagonalizability of H means the existence of a complete biorthonormal eigensystem $\{(\phi_n, \psi_n)\}$:

$$H\psi_n = E_n\psi_n, \quad H^\dagger\phi_n = E_n^*\phi_n, \quad \langle\phi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Qxn: What are the necessary and sufficient conditions for an operator to be diagonalizable and have a real spectrum.

Ans: It must satisfy $H^\dagger = \eta_+ H \eta_+{}^{-1}$ for some positive-definite metric operator η_+.

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Let $H : \mathscr{H} \to \mathscr{H}$ be an operator with a discrete spectrum.

Diagonalizability of H means the existence of a complete biorthonormal eigensystem $\{(\phi_n, \psi_n)\}$:

$$H\psi_n = E_n\psi_n, \quad H^\dagger\phi_n = E_n^*\phi_n, \quad \langle\phi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Qxn: What are the necessary and sufficient conditions for an operator to be diagonalizable and have a real spectrum.

Ans: It must satisfy $H^\dagger = \eta_+ H \eta_+{}^{-1}$ for some positive-definite metric operator η_+.

In this case it becomes Hermitian if we change the inner product of the Hilbert space:

$$\langle\psi|\phi\rangle \;\to\; \langle\psi,\phi\rangle_{\eta_+} := \langle\psi|\eta_+\phi\rangle$$

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Let $H : \mathscr{H} \to \mathscr{H}$ be an operator with a discrete spectrum.

Diagonalizability of H means the existence of a complete biorthonormal eigensystem $\{(\phi_n, \psi_n)\}$:

$$H\psi_n = E_n\psi_n, \quad H^\dagger\phi_n = E_n^*\phi_n, \quad \langle\phi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Qxn: What are the necessary and sufficient conditions for an operator to be diagonalizable and have a real spectrum.

Ans: It must satisfy $H^\dagger = \eta_+ H \eta_+{}^{-1}$ for some positive-definite metric operator η_+.

In this case it becomes Hermitian if we change the inner product of the Hilbert space:

$$\langle\psi|\phi\rangle \;\to\; \langle\psi,\phi\rangle_{\eta_+} := \langle\psi|\eta_+\phi\rangle \qquad \eta_+ = \sum_n |\phi_n\rangle\langle\phi_n|$$

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Let $H : \mathscr{H} \to \mathscr{H}$ be an operator with a discrete spectrum.

Diagonalizability of H means the existence of a complete biorthonormal eigensystem $\{(\phi_n, \psi_n)\}$:

$$H\psi_n = E_n\psi_n, \quad H^\dagger\phi_n = E_n^*\phi_n, \quad \langle\phi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Qxn: What are the necessary and sufficient conditions for an operator to be diagonalizable and have a real spectrum.

Ans: It must satisfy $H^\dagger = \eta_+ H \eta_+{}^{-1}$ for some positive-definite metric operator η_+.

In this case it becomes Hermitian if we change the inner product of the Hilbert space:

$$\langle\psi|\phi\rangle \;\to\; \langle\psi,\phi\rangle_{\eta_+} := \langle\psi|\eta_+\phi\rangle \qquad \eta_+ = \sum_n |\phi_n\rangle\langle\phi_n|$$

A.M., "Pseudo-Hermiticity versus $\mathcal{PT}$-Symmetry I, II, III," JMP **43**, 205, 2814, 3944 (2002).

Pseudo-Hermitian QM:

- Use $\langle\cdot,\cdot\rangle_{\eta_+}$ to construct a Hilbert space, $\mathscr{H}_{\eta_+}$.
- Because $H : \mathscr{H}_{\eta_+} \to \mathscr{H}_{\eta_+}$ is Hermitian, $(\mathscr{H}_{\eta_+}, H)$ defines a unitary quantum system.

Pseudo-Hermitian QM:

- Use $\langle\cdot,\cdot\rangle_{\eta_+}$ to construct a Hilbert space, $\mathscr{H}_{\eta_+}$.

- Because $H:\mathscr{H}_{\eta_+}\to\mathscr{H}_{\eta_+}$ is Hermitian, $(\mathscr{H}_{\eta_+},H)$ defines a unitary quantum system.

- The same system can be described by a Hermitian operator $h:\mathscr{H}\to\mathscr{H}$ which is typically a **nonlocal** operator,

$$h:=\eta_+^{\frac{1}{2}}H\eta_+^{-\frac{1}{2}}$$

A. M. Int. J. Geom. Meth. Mod. Phys. 7, 1191 (2010); **arXiv:0810.5643**.

Pseudo-Hermitian QM:

- Use $\langle\cdot,\cdot\rangle_{\eta_+}$ to construct a Hilbert space, $\mathscr{H}_{\eta_+}$.

- Because $H:\mathscr{H}_{\eta_+}\to\mathscr{H}_{\eta_+}$ is Hermitian, $(\mathscr{H}_{\eta_+},H)$ defines a unitary quantum system.

- The same system can be described by a Hermitian operator $h:\mathscr{H}\to\mathscr{H}$ which is typically a **nonlocal** operator,

$$h := \eta_+^{\frac{1}{2}} H \eta_+^{-\frac{1}{2}}$$

- The basic ingredient is the metric operator η_+.

A. M. Int. J. Geom. Meth. Mod. Phys. 7, 1191 (2010); **arXiv:0810.5643**.

Examples: $H=\frac{p^2}{2m}+\frac{\mu^2}{2}x^2+\mathrm{i}\epsilon x^3$ $\qquad p=-i\hbar\frac{d}{dx}$

$$h=\frac{p^2}{2m}+\frac{1}{2}\mu^2x^2+\frac{3}{2\mu^4}\left(\frac{1}{m}\{x^2,p^2\}+\mu^2x^4+\frac{2\hbar^2}{3m}\right)\epsilon^2+\frac{2}{\mu^{12}}\left(\frac{p^6}{m^3}-\frac{9\mu^2}{m^2}\{x^2,p^4\}\right.$$
$$\left.-\frac{51\mu^4}{8m}\{x^4,p^2\}-\frac{7\mu^6}{4}x^6-\frac{81\hbar^2\mu^2}{2m^2}p^2-\frac{69\hbar^2\mu^4}{2m}x^2\right)\epsilon^4+\mathcal{O}(\epsilon^6)$$

$$H=\frac{p^2}{2}+i\epsilon x^3$$

$$h=\frac{p^2}{2}+\frac{3}{16}\left(\left\{x^6,\frac{1}{p^2}\right\}+22\left\{x^4,\frac{1}{p^4}\right\}+(510+10\tilde{\lambda}_1)\left\{x^2,\frac{1}{p^6}\right\}\right.$$
$$\left.+\frac{8820+140\tilde{\lambda}_1}{p^8}-\frac{4}{3}\kappa_1\left\{x^3,\frac{1}{p^5}\right\}\mathcal{P}\right)\epsilon^2+\frac{1}{4}\left(15\lambda_2\left(\left\{x^2,\frac{1}{p^{11}}\right\}+\frac{44}{p^{13}}\right)\right.$$
$$\left.-\mathrm{i}\kappa_2\left\{x^3,\frac{1}{p^{10}}\right\}\mathcal{P}\right)\epsilon^3+\mathcal{O}(\epsilon^4).$$

$\tilde{\lambda}_1,\tilde{\lambda}_2,\kappa_1,\kappa_2\in\mathbb{R}$: η **is not unique.**

[JPA 38 (2005) 6557 & 39 (2006) 13495]

For $H = \begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix}$ with $x \in \mathbb{R}^+$: $E_\pm = \pm\, x$,

$x = 0$ is an **exceptional point** (EP).

For $H = \begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix}$ with $x \in \mathbb{R}^+$: $E_\pm = \pm\, x$,

$x = 0$ is an **exceptional point** (EP).

$$\psi_\pm = \frac{N_\pm}{\sqrt{2}} \begin{pmatrix} \pm 1 \\ x \end{pmatrix}, \quad \phi_\pm = \frac{1}{\sqrt{2} N_\pm^*} \begin{pmatrix} \pm 1 \\ x^{-1} \end{pmatrix}$$

For $H = \begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix}$ with $x \in \mathbb{R}^+$: $E_\pm = \pm x$,

$x = 0$ is an **exceptional point** (EP).

$$\psi_\pm = \frac{N_\pm}{\sqrt{2}} \begin{pmatrix} \pm 1 \\ x \end{pmatrix}, \quad \phi_\pm = \frac{1}{\sqrt{2} N_\pm^*} \begin{pmatrix} \pm 1 \\ x^{-1} \end{pmatrix}$$

No complete biorthonormal eigensystem exists at an EP.

For $H = \begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix}$ with $x \in \mathbb{R}^+$: $E_\pm = \pm x$,

$x = 0$ is an **exceptional point** (EP).

$$\psi_\pm = \frac{N_\pm}{\sqrt{2}} \begin{pmatrix} \pm 1 \\ x \end{pmatrix}, \qquad \phi_\pm = \frac{1}{\sqrt{2} N_\pm^*} \begin{pmatrix} \pm 1 \\ x^{-1} \end{pmatrix}$$

No complete biorthonormal eigensystem exists at an EP.

$$\eta_+ = \begin{pmatrix} a & b\, x^{-1} \\ b\, x^{-1} & a\, x^{-2} \end{pmatrix}$$

$$a, b \in \mathbb{R}, \ a > 0, \ a \pm b > 0$$

EPs are singularities of the metric operator.

Spectral Singularities

- $H = -\dfrac{d^2}{dx^2} + v(x)$
- $v : \mathbb{R} \to \mathbb{C}$ is a complex scattering potential:

$$v(x) \to 0, \quad \text{as} \quad |x| \to \infty$$

- Spec(H)= $[0, \infty)$

Spectral Singularities

- $H = -\dfrac{d^2}{dx^2} + v(x)$
- $v : \mathbb{R} \to \mathbb{C}$ is a complex scattering potential:

$$v(x) \to 0, \quad \text{as} \quad |x| \to \infty$$

- Spec(H)$= [0, \infty)$
- H is called diagonalizable if there is complete biorthonormal system of eigenfunctions of H and $H^\dagger$:

$$H\psi_{k,a} = k^2\psi_{k,a}, \qquad H^\dagger\phi_{k,a} = k^2\phi_{k,a}, \qquad a = 1, 2$$

$$\langle\phi_{k,a}|\psi_{q,b}\rangle = \delta(k-q)\delta_{ab}, \quad \int_0^\infty dk \sum_{a=1}^{2} |\psi_{k,a}\rangle\langle\phi_{k,a}| = 1$$

$$\eta_+ = \int_0^\infty dk \sum_{a=1}^{2} |\phi_{k,a}\rangle\langle\phi_{k,a}|$$

Spectral Singularities

- $H = -\dfrac{d^2}{dx^2} + v(x)$
- $v : \mathbb{R} \to \mathbb{C}$ is a complex scattering potential:

$$v(x) \to 0, \quad \text{as} \quad |x| \to \infty$$

- Spec(H)$= [0, \infty)$
- H is called diagonalizable if there is complete biorthonormal system of eigenfunctions of H and $H^\dagger$:

$$H\psi_{k,a} = k^2\psi_{k,a}, \qquad H^\dagger\phi_{k,a} = k^2\phi_{k,a}, \qquad a = 1,2$$

$$\langle\phi_{k,a}|\psi_{q,b}\rangle = \delta(k-q)\delta_{ab}, \quad \int_0^\infty dk \sum_{a=1}^{2} |\psi_{k,a}\rangle\langle\phi_{k,a}| = 1$$

$$\eta_+ = \int_0^\infty dk \sum_{a=1}^{2} |\phi_{k,a}\rangle\langle\phi_{k,a}|$$

- v is an analytic function of complex coupling constants z.

$$v(x) \to v^z(x), \qquad \psi_{k,a}(x) \to \psi^z_{k,a}(x)$$

- Let $\langle\psi^{z^*}_{k,a}|\psi^{z}_{q,b}\rangle =: K_{ab}\,\delta(k-q)$ & $\mathbf{K} := (K_{ab})$.

- Let $\langle\psi^{z^*}_{k,a}|\psi^{z}_{q,b}\rangle =: K_{ab}\,\delta(k-q)$ & $\mathbf{K} := (K_{ab})$. Then,

$$|\phi^{z}_{a,k}\rangle = \sum_{b=1}^{2} (\mathbf{K}^{-1*})_{ab}|\psi^{z^*}_{b,z}\rangle$$

$$\eta_+ = \int_0^\infty dk \sum_{b,c=1}^{2} \left(\mathbf{K}^{-1\dagger}\mathbf{K}^{-1}\right)_{bc} |\psi^{z^*}_{b,k}\rangle\langle\psi^{z^*}_{c,k}|$$

[A. M. & H. Mehri-Dehnavi, JPA 42 (2009) 125303]

- Let $\langle\psi^{z^*}_{k,a}|\psi^{z}_{q,b}\rangle =: K_{ab}\,\delta(k-q)$ & $\mathbf{K} := (K_{ab})$. Then,

$$|\phi^{z}_{a,k}\rangle = \sum_{b=1}^{2}(\mathbf{K}^{-1*})_{ab}|\psi^{z^*}_{b,z}\rangle$$

$$\eta_+ = \int_0^\infty dk \sum_{b,c=1}^{2}\left(\mathbf{K}^{-1\dagger}\mathbf{K}^{-1}\right)_{bc}|\psi^{z^*}_{b,k}\rangle\langle\psi^{z^*}_{c,k}|$$

$\det\mathbf{K}(k) = 0 \iff k^2$ **is a spectral singularity**.

[A. M. & H. Mehri-Dehnavi, JPA 42 (2009) 125303]

- Let $\langle\psi^{z^*}_{k,a}|\psi^{z}_{q,b}\rangle =: K_{ab}\,\delta(k-q)$ & $\mathbf{K} := (K_{ab})$. Then,

$$|\phi^{z}_{a,k}\rangle = \sum_{b=1}^{2}(\mathbf{K}^{-1*})_{ab}|\psi^{z^*}_{b,z}\rangle$$

$$\eta_+ = \int_0^\infty dk \sum_{b,c=1}^{2}\left(\mathbf{K}^{-1\dagger}\mathbf{K}^{-1}\right)_{bc}|\psi^{z^*}_{b,k}\rangle\langle\psi^{z^*}_{c,k}|$$

$\det\mathbf{K}(k) = 0 \iff k^2$ **is a spectral singularity.**

$$\Longrightarrow \begin{cases} H \text{ is not diagonalizable.} \\ \text{No metric operator } \eta_+ \text{ satisfying } H^\dagger = \eta_+ H \eta_+{}^{-1}. \end{cases}$$

[A. M. & H. Mehri-Dehnavi, JPA 42 (2009) 125303]

- Let $\langle\psi^{z^*}_{k,a}|\psi^{z}_{q,b}\rangle =: K_{ab}\,\delta(k-q)$ & $\mathbf{K} := (K_{ab})$. Then,

$$|\phi^{z}_{a,k}\rangle = \sum_{b=1}^{2} (\mathbf{K}^{-1*})_{ab}|\psi^{z^*}_{b,z}\rangle$$

$$\eta_+ = \int_0^\infty dk \sum_{b,c=1}^{2} \left(\mathbf{K}^{-1\dagger}\mathbf{K}^{-1}\right)_{bc} |\psi^{z^*}_{b,k}\rangle\langle\psi^{z^*}_{c,k}|$$

$\det \mathbf{K}(k) = 0 \iff k^2$ **is a spectral singularity.**

$$\Longrightarrow \begin{cases} H \text{ is not diagonalizable.} \\ \text{No metric operator } \eta_+ \text{ satisfying } H^\dagger = \eta_+ H \eta_+{}^{-1}. \end{cases}$$

Spectral singularities are also singularities of the metric operator.

[A. M. & H. Mehri-Dehnavi, JPA 42 (2009) 125303]

- Asymptotic solutions:

$$\psi(x) = A_{\pm}e^{ikx} + B_{\pm}e^{-ikx} \quad \text{for} \quad x \to \pm\infty.$$

- Transfer matrix: $\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \begin{bmatrix} M_{11}(k) & M_{12}(k) \\ M_{21}(k) & M_{22}(k) \end{bmatrix} \begin{bmatrix} A_- \\ B_- \end{bmatrix}.$

- Asymptotic solutions:

$$\psi(x) = A_\pm e^{ikx} + B_\pm e^{-ikx} \quad \text{for} \quad x \to \pm\infty.$$

- Transfer matrix: $\left[\begin{array}{c} A_+ \\ B_+ \end{array}\right] = \left[\begin{array}{cc} M_{11}(k) & M_{12}(k) \\ M_{21}(k) & M_{22}(k) \end{array}\right] \left[\begin{array}{c} A_- \\ B_- \end{array}\right].$

- Spectral singularities are real zeros of $M_{22}(k)$.

- Asymptotic solutions:

$$\psi(x) = A_\pm e^{ikx} + B_\pm e^{-ikx} \quad \text{for} \quad x \to \pm\infty.$$

- Transfer matrix: $\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \begin{bmatrix} M_{11}(k) & M_{12}(k) \\ M_{21}(k) & M_{22}(k) \end{bmatrix} \begin{bmatrix} A_- \\ B_- \end{bmatrix}.$

- Spectral singularities are real zeros of $M_{22}(k)$.

- Example: $v(x) = z\,\delta(x)$, $z \in \mathbb{C}$:

  - Transfer matrix: $\mathbf{M} = \begin{bmatrix} 1 - \frac{iz}{2k} & -\frac{iz}{2k} \\ \frac{iz}{2k} & 1 + \frac{iz}{2k} \end{bmatrix}$

  - There is a spectral singularity for $z \in i\mathbb{R}$ at $k^2 = -\dfrac{z^2}{4}$.

[A. M., JPA 39 (2006) 13506]

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$T^l = T^r =: T$$

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$T^l = T^r =: T$$

$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T = \frac{1}{M_{22}}$$

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$T^l = T^r =: T$$

$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T = \frac{1}{M_{22}}$$

- Reflection and Transmission amplitudes diverge at a spectral singularity.

$\Rightarrow$ infinite amplification of incident waves.

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \left\{ \begin{array}{cl} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{array} \right.$$

$$\psi^{\text{right}}(x) = \left\{ \begin{array}{cl} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{array} \right.$$

$$T^l = T^r =: T$$

$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T = \frac{1}{M_{22}}$$

- Reflection and Transmission amplitudes diverge at a spectral singularity.

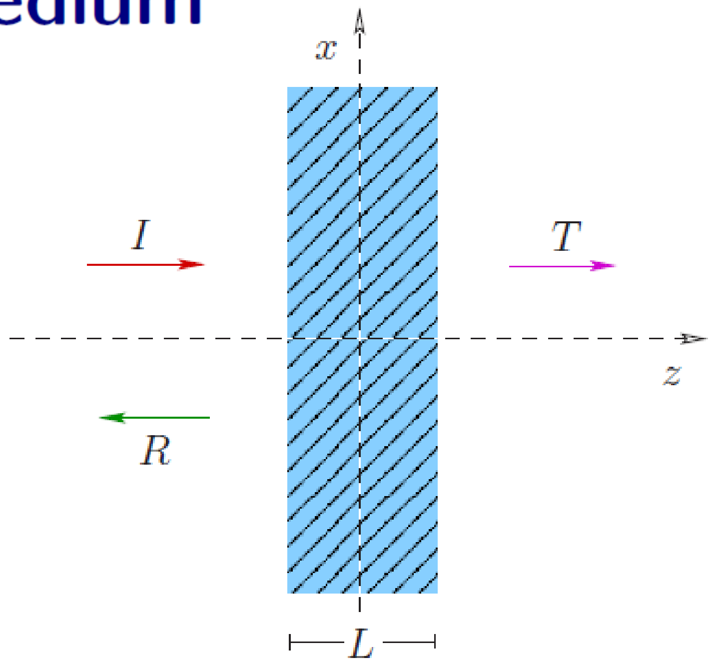
$\Rightarrow$ infinite amplification of incident waves.

- Physically they correspond to scattering states that behave like resonances: Zero-width resonances.

[PRL **102**, 220402 (2009); arXiv:0901.4472]

A Physical Application:

Infinite planar slab gain medium



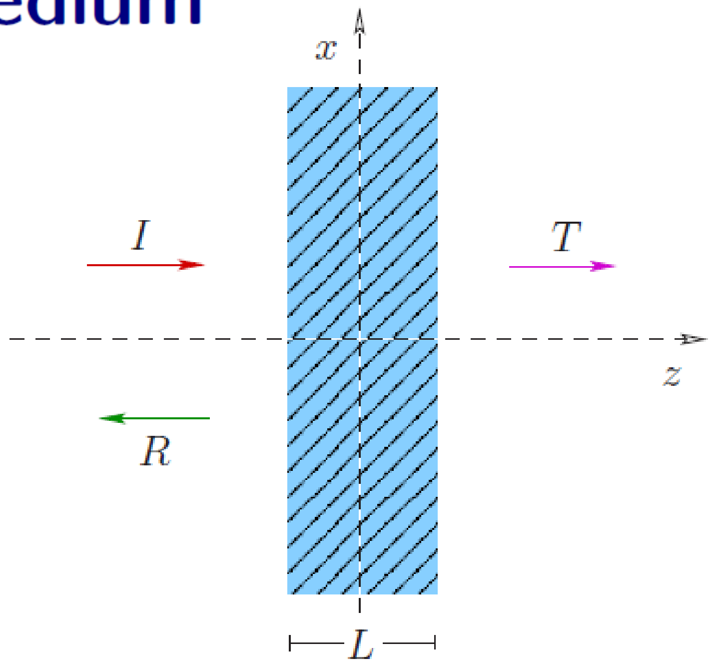
Maxwell's equations:

$$\nabla \cdot (\mathfrak{n}^2\, \vec{E}) = 0,$$
$$\nabla \cdot \vec{B} = 0,$$
$$\partial_t \vec{B} = -\nabla \times \vec{E},$$
$$\mathfrak{n}^2\, \partial_t \vec{E} = c^2 \nabla \times \vec{E}.$$

$\mathfrak{n} := \eta + i\kappa$: Refractive index

A Physical Application:

Infinite planar slab gain medium



Maxwell's equations:

$$\nabla \cdot (\mathfrak{n}^2\, \vec{E}) = 0,$$
$$\nabla \cdot \vec{B} = 0,$$
$$\partial_t \vec{B} = -\nabla \times \vec{E},$$
$$\mathfrak{n}^2\, \partial_t \vec{E} = c^2 \nabla \times \vec{E}.$$

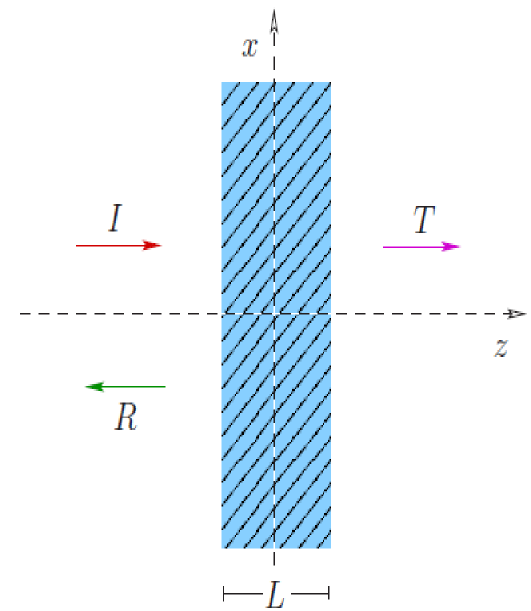
$\mathfrak{n} := \eta + i\kappa$: Refractive index

$$\mathfrak{n}^2 \partial_t^2 \vec{E} - c^2 \partial_z^2 \vec{E} = 0$$
$$\vec{E} \propto e^{\frac{i\omega}{c}(\mathfrak{n}z - ct)}\, \hat{\imath} = e^{-\frac{\omega\kappa z}{c}}\, e^{\frac{i\omega}{c}(\eta z - ct)}\, \hat{\imath}$$

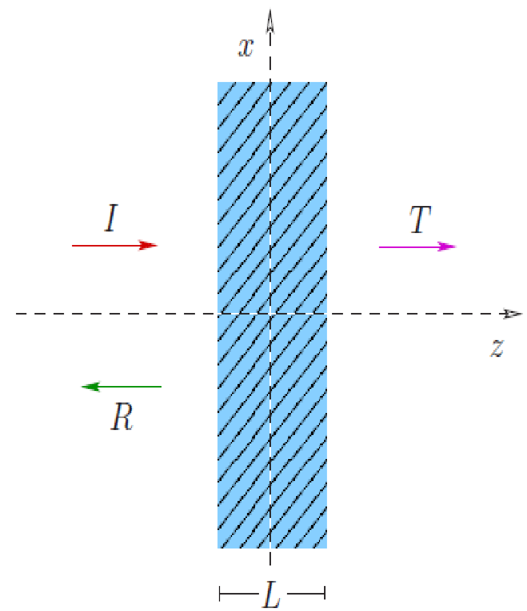
$|\vec{E}|^2$ grows exponentially for $\kappa < 0$: Gain Medium

Gain Coefficient: $g := -\frac{2\omega\kappa}{c}$

$$\vec{E}(z,t) = E\,e^{-i\omega t}\psi(z)\,\hat{\imath},$$
$$\vec{B}(z,t) = -i\omega^{-1}E\,e^{-i\omega t}\psi'(z)\,\hat{\jmath}$$



$$\vec{E}(z,t) = E\, e^{-i\omega t}\psi(z)\,\hat{\imath},$$
$$\vec{B}(z,t) = -i\omega^{-1}E\, e^{-i\omega t}\psi'(z)\,\hat{\jmath}$$



$$-\psi''(z) + v(z)\psi(z) = k^2\psi(z)$$

Complex Barrier Potential:

$$v(z) := \begin{cases} \mathfrak{z} & \text{for} \quad |z| \leq L/2 \\ 0 & \text{for} \quad |z| > L/2 \end{cases}$$

$$\mathfrak{z} := k^2(1-\mathfrak{n}^2) \in \mathbb{C}, \qquad k := \omega/c$$

$\mathfrak{n} := \eta + i\kappa$: Refractive index

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Taking Absolute value of both sides gives:

$$e^{-2gL} = |\mathcal{R}|^2 \quad \Rightarrow \quad g = \frac{1}{2L} \ln \frac{1}{|\mathcal{R}|^2}$$

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\tfrac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\tfrac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Taking Absolute value of both sides gives:

$$e^{-2gL} = |\mathcal{R}|^2 \quad \Rightarrow \quad g = \frac{1}{2L} \ln \frac{1}{|\mathcal{R}|^2}$$

This is known as the Laser Threshold Condition.

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Taking Absolute value of both sides gives:

$$e^{-2gL} = |\mathcal{R}|^2 \quad \Rightarrow \quad g = \frac{1}{2L} \ln \frac{1}{|\mathcal{R}|^2}$$

This is known as the Laser Threshold Condition.

Optical Spectral Singularity $\Rightarrow$ **Lasing at threshold gain**

[A. M. PRA **83**, 045801 (2011); arXiv:1102.4695]

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\tfrac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\tfrac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Taking Absolute value of both sides gives:

$$e^{-2gL} = |\mathcal{R}|^2 \quad \Rightarrow \quad g = \frac{1}{2L} \ln \frac{1}{|\mathcal{R}|^2}$$

This is known as the Laser Threshold Condition.

Optical Spectral Singularity $\Rightarrow$ **Lasing at threshold gain**

[A. M. PRA **83**, 045801 (2011); arXiv:1102.4695]

- Time-reversal transformation: $\mathbf{M} \xrightarrow{\mathcal{T}} \sigma_1 \mathbf{M}^* \sigma_1$

$$\Rightarrow M_{22} \xrightarrow{\mathcal{T}} M_{11}^* \text{ and } M_{11} \xrightarrow{\mathcal{T}} M_{22}^*.$$

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Taking Absolute value of both sides gives:

$$e^{-2gL} = |\mathcal{R}|^2 \quad \Rightarrow \quad g = \frac{1}{2L} \ln \frac{1}{|\mathcal{R}|^2}$$

This is known as the Laser Threshold Condition.

Optical Spectral Singularity $\Rightarrow$ **Lasing at threshold gain**

[A. M. PRA **83**, 045801 (2011); arXiv:1102.4695]

- Time-reversal transformation: $\mathbf{M} \xrightarrow{\mathcal{T}} \sigma_1 \mathbf{M}^* \sigma_1$

$$\Rightarrow M_{22} \xrightarrow{\mathcal{T}} M_{11}^* \text{ and } M_{11} \xrightarrow{\mathcal{T}} M_{22}^*.$$

- Time-reversed optical spectral singularities:

$\Rightarrow$ Antilasing (CPA) [Wan et al Science 2010]

Parity transformation: $\mathbf{M} \xrightarrow{\mathcal{P}} \sigma_1 \mathbf{M}^{-1} \sigma_1 \Rightarrow M_{22} \xrightarrow{\mathcal{P}} M_{22}$

spectrally sing. potentials $\xrightarrow{\mathcal{P}}$ spectrally sing. potentials

Parity transformation: $\mathbf{M} \xrightarrow{\mathcal{P}} \sigma_1 \mathbf{M}^{-1} \sigma_1 \Rightarrow M_{22} \xrightarrow{\mathcal{P}} M_{22}$

spectrally sing. potentials $\xrightarrow{\mathcal{P}}$ spectrally sing. potentials

$\mathcal{PT}$-transformation: $\mathbf{M} \xrightarrow{\mathcal{PT}} \mathbf{M}^{-1*}$

$$\Rightarrow M_{11} \xrightarrow{\mathcal{PT}} M_{22}^* \text{ and } M_{22} \xrightarrow{\mathcal{PT}} M_{11}^*$$

For a $\mathcal{PT}$-symmetric potential: $M_{22}(k) = 0 \Leftrightarrow M_{11}(k) = 0$

Parity transformation: $\mathbf{M} \xrightarrow{\mathcal{P}} \sigma_1 \mathbf{M}^{-1}\sigma_1 \Rightarrow M_{22} \xrightarrow{\mathcal{P}} M_{22}$

spectrally sing. potentials $\xrightarrow{\mathcal{P}}$ spectrally sing. potentials

$\mathcal{PT}$-transformation: $\mathbf{M} \xrightarrow{\mathcal{PT}} \mathbf{M}^{-1*}$

$$\Rightarrow M_{11} \xrightarrow{\mathcal{PT}} M_{22}^{*} \text{ and } M_{22} \xrightarrow{\mathcal{PT}} M_{11}^{*}$$

For a $\mathcal{PT}$-symmetric potential: $M_{22}(k) = 0 \Leftrightarrow M_{11}(k) = 0$

Spectral singularities of $\mathcal{PT}$-symmetric potential accompany their $\mathcal{T}$-dual; they are self-dual spectral singularities.

Parity transformation: $\mathbf{M} \xrightarrow{\mathcal{P}} \sigma_1 \mathbf{M}^{-1} \sigma_1 \Rightarrow M_{22} \xrightarrow{\mathcal{P}} M_{22}$

spectrally sing. potentials $\xrightarrow{\mathcal{P}}$ spectrally sing. potentials

$\mathcal{PT}$-transformation: $\mathbf{M} \xrightarrow{\mathcal{PT}} \mathbf{M}^{-1*}$

$$\Rightarrow M_{11} \xrightarrow{\mathcal{PT}} M_{22}^* \text{ and } M_{22} \xrightarrow{\mathcal{PT}} M_{11}^*$$

For a $\mathcal{PT}$-symmetric potential: $M_{22}(k) = 0 \Leftrightarrow M_{11}(k) = 0$

Spectral singularities of $\mathcal{PT}$-symmetric potential accompany their $\mathcal{T}$-dual; they are self-dual spectral singularities.

$\mathcal{PT}$-symmetric optical systems: OSS $\Leftrightarrow$ CPA

$\Rightarrow$ CPA-lasers

Parity transformation: $\mathbf{M} \xrightarrow{\mathcal{P}} \sigma_1 \mathbf{M}^{-1} \sigma_1 \Rightarrow M_{22} \xrightarrow{\mathcal{P}} M_{22}$

spectrally sing. potentials $\xrightarrow{\mathcal{P}}$ spectrally sing. potentials

$\mathcal{PT}$-transformation: $\mathbf{M} \xrightarrow{\mathcal{PT}} \mathbf{M}^{-1*}$

$$\Rightarrow M_{11} \xrightarrow{\mathcal{PT}} M_{22}^* \text{ and } M_{22} \xrightarrow{\mathcal{PT}} M_{11}^*$$

For a $\mathcal{PT}$-symmetric potential: $M_{22}(k) = 0 \Leftrightarrow M_{11}(k) = 0$

Spectral singularities of $\mathcal{PT}$-symmetric potential accompany their $\mathcal{T}$-dual; they are self-dual spectral singularities.

$\mathcal{PT}$-symmetric optical systems: OSS $\Leftrightarrow$ CPA

$\Rightarrow$ CPA-lasers

Self-dual spectral singularities are given by real values of k such that $M_{11}(k) = 0$ and $M_{22}(k) = 0$

$\mathcal{P}$ and $\mathcal{T}$ map potentials with a self-dual spectral singularities k^2 to potentials with a self-dual spectral singularities k^2.

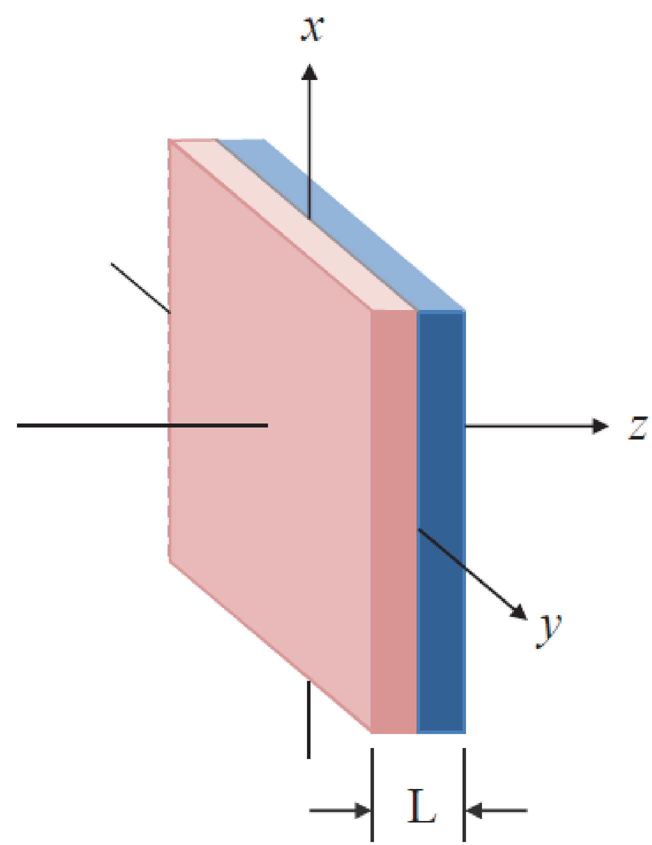
There are non-$\mathcal{PT}$-symmetric potentials with self-dual spectral singularities:

$\Rightarrow$ Non-$\mathcal{PT}$-symmetric CPA-lasers

There are non-$\mathcal{PT}$-symmetric potentials with self-dual spectral singularities:

$\Rightarrow$ Non-$\mathcal{PT}$-symmetric CPA-lasers

$$\mathfrak{n}(\mathrm{z}) := \begin{cases} \mathfrak{n}_1 & \text{for} \quad -\frac{L}{2} \leq \mathrm{z} < 0, \\ \mathfrak{n}_2 & \text{for} \quad 0 \leq \mathrm{z} \leq \frac{L}{2} \\ 1 & \text{for} \quad |\mathrm{z}| > \frac{L}{2}. \end{cases}$$



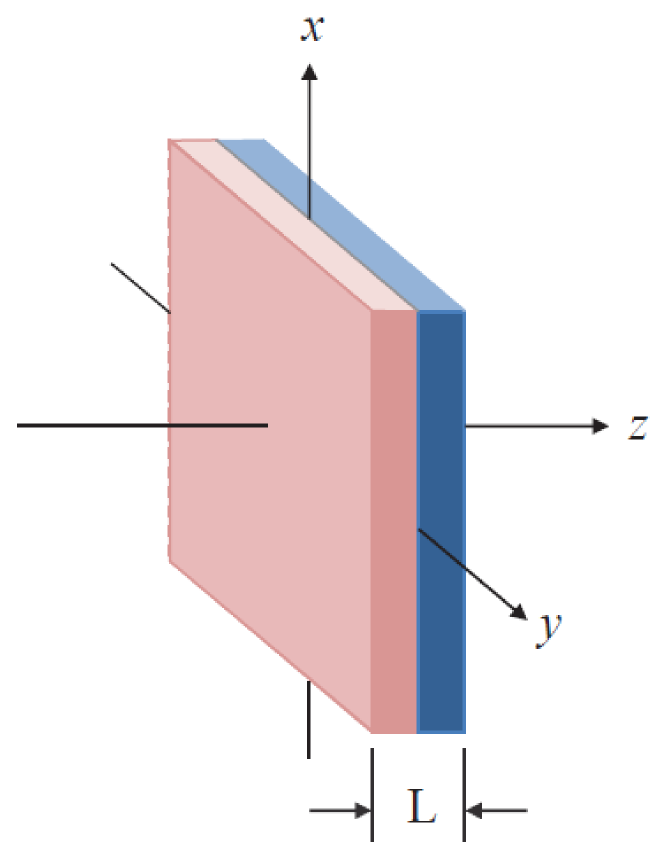
There are non-$\mathcal{PT}$-symmetric potentials with self-dual spectral singularities:

$\Rightarrow$ Non-$\mathcal{PT}$-symmetric CPA-lasers

$$\mathfrak{n}(\mathrm{z}) := \begin{cases} \mathfrak{n}_1 & \text{for} \quad -\frac{L}{2} \leq \mathrm{z} < 0, \\ \mathfrak{n}_2 & \text{for} \quad 0 \leq \mathrm{z} \leq \frac{L}{2} \\ 1 & \text{for} \quad |\mathrm{z}| > \frac{L}{2}. \end{cases}$$

$L/\lambda = 200$

$\mathfrak{n}_1$	$3.603 \pm 1.178 \times 10^{-3} i$
$\mathfrak{n}_2$	$1.498 \mp 2.243 \times 10^{-3} i$
$\mathfrak{n}_1$	$3.600 \pm 2.520 \times 10^{-3} i$
$\mathfrak{n}_2$	$2.997 \mp 2.695 \times 10^{-3} i$
$\mathfrak{n}_1$	$3.000 \pm 1.370 \times 10^{-3} i$
$\mathfrak{n}_2$	$1.398 \mp 2.431 \times 10^{-3} i$



[JPA **45**, 444024 (2012); arXiv:1205.4560]

Unidirectional Invisibility & $\mathcal{PT}$-Symmetry

Unidir. Reflectionlessness: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$

Unidir. Invisibility: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$ & $T = 1$

Unidirectional Invisibility & $\mathcal{PT}$-Symmetry

Unidir. Reflectionlessness: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$

Unidir. Invisibility: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$ & $T = 1$

Lin et al PRL **106**, 213901 (2011): Unidir. invisibility in the $\mathcal{PT}$-symmetric locally periodic potential

$$v(x) := \begin{cases} \alpha_0 + \alpha\, e^{2i\beta x} & \text{for} \quad |x| \leq \frac{L}{2}, \\ 0 & \text{for} \quad |x| > \frac{L}{2}, \end{cases} \qquad \alpha_0, \alpha, \beta \in \mathbb{R}$$

Unidirectional Invisibility & $\mathcal{PT}$-Symmetry

Unidir. Reflectionlessness: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$

Unidir. Invisibility: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$ & $T = 1$

Lin et al PRL **106**, 213901 (2011): Unidir. invisibility in the $\mathcal{PT}$-symmetric locally periodic potential

$$v(x) := \begin{cases} \alpha_0 + \alpha\, e^{2i\beta x} & \text{for} \quad |x| \leq \frac{L}{2}, \\ 0 & \text{for} \quad |x| > \frac{L}{2}, \end{cases} \qquad \alpha_0, \alpha, \beta \in \mathbb{R}$$

Gasymov, Func. Anal. Appl. **44**, 11 (1980).

Curtright & Mezincescu, JMP **48**, 092106 (2007).

Midya et al, PLA **374**, 2605 (2010).

Graefe & Jones, PRA **84**, 013818 (2011).

Longhi, JPA **44**, 485302 (2011).

Jones, JPA **45**, 135306 (2012).

Uzdin & Moiseyev, PRA **85**, 031804 (2012).

Unidirectional Invisibility & $\mathcal{PT}$-Symmetry

Unidir. Reflectionlessness: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$

Unidir. Invisibility:

- $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$ & $T = 1$

$$\mathbf{M} = \begin{bmatrix} T - \dfrac{R^l R^r}{T} & \dfrac{R^r}{T} \\ -\dfrac{R^l}{T} & \dfrac{1}{T} \end{bmatrix}$$

- $M_{12} \neq 0 = M_{21}$ or $M_{21} \neq 0 = M_{12}$ & $M_{11} = M_{22} = 1$

Unidirectional Invisibility & $\mathcal{PT}$-Symmetry

Unidir. Reflectionlessness: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$

Unidir. Invisibility:

- $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$ & $T = 1$

$$\mathbf{M} = \begin{bmatrix} T - \dfrac{R^l R^r}{T} & \dfrac{R^r}{T} \\ -\dfrac{R^l}{T} & \dfrac{1}{T} \end{bmatrix}$$

- $M_{12} \neq 0 = M_{21}$ or $M_{21} \neq 0 = M_{12}$ & $M_{11} = M_{22} = 1$

Under $\mathcal{PT}$-transformation: $\mathbf{M} \xrightarrow{\mathcal{PT}} \mathbf{M}^{-1*}$ or

$$M_{11} \xrightarrow{\mathcal{PT}} M_{22}^*, \qquad M_{12} \xrightarrow{\mathcal{PT}} -M_{12}^*,$$
$$M_{21} \xrightarrow{\mathcal{PT}} -M_{21}^*, \qquad M_{22} \xrightarrow{\mathcal{PT}} M_{11}^*.$$

Unidirectional Invisibility & $\mathcal{PT}$-Symmetry

Unidir. Reflectionlessness: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$

Unidir. Invisibility:

- $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$ & $T = 1$

$$\mathbf{M} = \begin{bmatrix} T - \dfrac{R^l R^r}{T} & \dfrac{R^r}{T} \\ -\dfrac{R^l}{T} & \dfrac{1}{T} \end{bmatrix}$$

- $M_{12} \neq 0 = M_{21}$ or $M_{21} \neq 0 = M_{12}$ & $M_{11} = M_{22} = 1$

Under $\mathcal{PT}$-transformation: $\mathbf{M} \xrightarrow{\mathcal{PT}} \mathbf{M}^{-1*}$ or

$$M_{11} \xrightarrow{\mathcal{PT}} M_{22}^*, \qquad M_{12} \xrightarrow{\mathcal{PT}} -M_{12}^*,$$
$$M_{21} \xrightarrow{\mathcal{PT}} -M_{21}^*, \qquad M_{22} \xrightarrow{\mathcal{PT}} M_{11}^*.$$

$\mathcal{PT}$ leaves the equations of unidirectional reflectionlessness & invisibility invariant.

$\mathcal{PT}$-invariance of the invisibility equations implies:

Theorem: The following equivalent statements hold.

- $v(x)$ is invisible from the left (or right) for $k = k_\star$ **iff** so is $v(-x)^*$.
- $v(x)$ is invisible from the left (resp. right) for $k = k_\star$ **iff** $v(x)^*$ is invisible from the right (resp. left) for $k = k_\star$.

$\mathcal{PT}$-invariance of the invisibility equations implies:

Theorem: The following equivalent statements hold.

- $v(x)$ is invisible from the left (or right) for $k = k_\star$ **iff** so is $v(-x)^*$.
- $v(x)$ is invisible from the left (resp. right) for $k = k_\star$ **iff** $v(x)^*$ is invisible from the right (resp. left) for $k = k_\star$.

Therefore, $\mathcal{P}$-symmetric (even) and $\mathcal{T}$-symmetric (real) potentials **do not support** unidirectional invisibility.

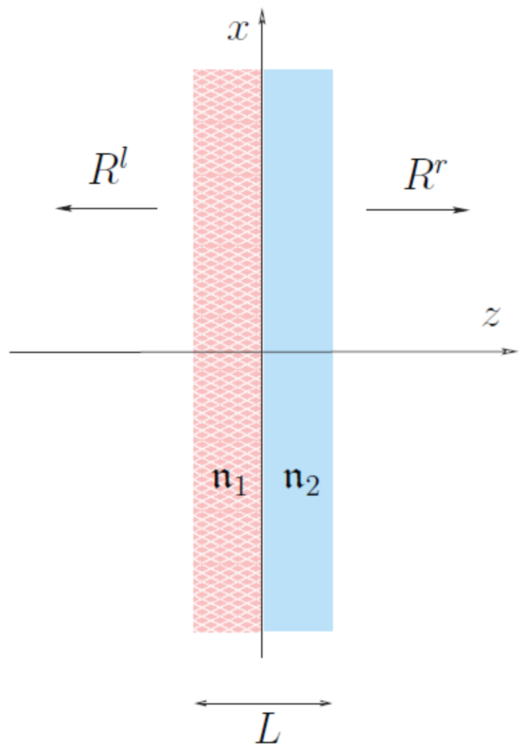
$\mathcal{PT}$**-symmetric** potentials **can support** unidirectional invisibility, but **so are** a large number of **non-**$\mathcal{PT}$**-symmetric** potentials.

A. M., PRA **87**, 012103 (2013), arXiv:1206.0116

Unidirectional Invisibility for a Two-layer slab

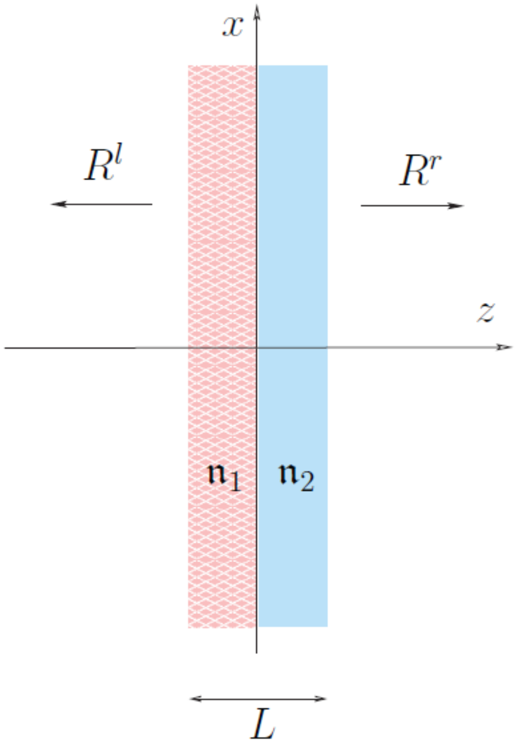
$$\begin{cases} \mathfrak{a}_+^2(\cos\mathfrak{a}_+ - \cos\mathfrak{K}) - \mathfrak{a}_-^2(\cos\mathfrak{a}_- - \cos\mathfrak{K}) = 0, \\ (\mathfrak{a}_+^2 - \mathfrak{a}_-^2 - \mathfrak{K}^2)\mathfrak{a}_+\sin\mathfrak{a}_+ \\ \qquad\qquad +(\mathfrak{a}_+^2 - \mathfrak{a}_-^2 + \mathfrak{K}^2)\mathfrak{a}_-\sin\mathfrak{a}_- \\ \qquad\qquad\qquad\qquad -2(\mathfrak{a}_+^2 - \mathfrak{a}_-^2)\mathfrak{K}\sin\mathfrak{K} = 0, \\ \cos\mathfrak{a}_+ \neq \cos\mathfrak{a}_-. \end{cases}$$

$$\mathfrak{a}_\pm := \frac{Lk(\mathfrak{n}_1 \pm \mathfrak{n}_2)}{2}, \qquad \mathfrak{K} := Lk$$



Unidirectional Invisibility for a Two-layer slab

$$\begin{cases} \mathfrak{a}_+^2(\cos\mathfrak{a}_+ - \cos\mathfrak{K}) - \mathfrak{a}_-^2(\cos\mathfrak{a}_- - \cos\mathfrak{K}) = 0, \\ (\mathfrak{a}_+^2 - \mathfrak{a}_-^2 - \mathfrak{K}^2)\mathfrak{a}_+\sin\mathfrak{a}_+ \\ \qquad +(\mathfrak{a}_+^2 - \mathfrak{a}_-^2 + \mathfrak{K}^2)\mathfrak{a}_-\sin\mathfrak{a}_- \\ \qquad\qquad -2(\mathfrak{a}_+^2 - \mathfrak{a}_-^2)\mathfrak{K}\sin\mathfrak{K} = 0, \\ \cos\mathfrak{a}_+ \neq \cos\mathfrak{a}_-. \end{cases}$$



$$\mathfrak{a}_\pm := \frac{Lk(\mathfrak{n}_1 \pm \mathfrak{n}_2)}{2}, \qquad \mathfrak{K} := Lk$$

These are invariant under $\mathcal{PT}$-transformation:

$$\mathfrak{a}_+ \xrightarrow{\mathcal{PT}} \mathfrak{a}_+^*, \qquad \mathfrak{a}_- \xrightarrow{\mathcal{PT}} -\mathfrak{a}_-^*$$

$$\mathfrak{a}_+^2(\cos\mathfrak{a}_+ - \cos\mathfrak{K}) - \mathfrak{a}_-^2(\cos\mathfrak{a}_- - \cos\mathfrak{K}) = 0,$$

$$\begin{aligned}(\mathfrak{a}_+^2 - \mathfrak{a}_-^2 - \mathfrak{K}^2)\mathfrak{a}_+ \sin\mathfrak{a}_+ \\ +(\mathfrak{a}_+^2 - \mathfrak{a}_-^2 + \mathfrak{K}^2)\mathfrak{a}_- \sin\mathfrak{a}_- \\ -2(\mathfrak{a}_+^2 - \mathfrak{a}_-^2)\mathfrak{K}\sin\mathfrak{K} = 0,\end{aligned}$$

$$\begin{aligned} x_\pm &:= \Re(\mathfrak{a}_\pm), \\ y_\pm &:= \Im(\mathfrak{a}_\pm)\end{aligned}$$

$$
\begin{aligned}
&\mathfrak{a}_+^2(\cos\mathfrak{a}_+ - \cos\mathfrak{K}) - \mathfrak{a}_-^2(\cos\mathfrak{a}_- - \cos\mathfrak{K}) = 0,\\
&(\mathfrak{a}_+^2 - \mathfrak{a}_-^2 - \mathfrak{K}^2)\mathfrak{a}_+\sin\mathfrak{a}_+\\
&\qquad +(\mathfrak{a}_+^2 - \mathfrak{a}_-^2 + \mathfrak{K}^2)\mathfrak{a}_-\sin\mathfrak{a}_-\\
&\qquad\qquad -2(\mathfrak{a}_+^2 - \mathfrak{a}_-^2)\mathfrak{K}\sin\mathfrak{K} = 0,
\end{aligned}
$$

$$
\begin{aligned}
x_\pm &:= \Re(\mathfrak{a}_\pm),\\
y_\pm &:= \Im(\mathfrak{a}_\pm)
\end{aligned}
$$

$$
\begin{aligned}
&(x_-^2 - y_-^2)\cos x_-\cosh y_- - (x_+^2 - y_+^2)\cos x_+\cosh y_+\\
&+2x_-y_-\sin x_-\sinh y_- - 2x_+y_+\sin x_+\sinh y_+ = (x_-^2 - x_+^2 - y_-^2 + y_+^2)\cos\mathfrak{K},
\end{aligned}
$$

$$
\begin{aligned}
&2x_-y_-\cos x_-\cosh y_- - 2x_+y_+\cos x_+\cosh y_+\\
&-(x_-^2 - y_-^2)\sin x_-\sinh y_- + (x_+^2 - y_+^2)\sin x_+\sinh y_+ = 2(x_-y_- - x_+y_+)\cos\mathfrak{K},
\end{aligned}
$$

$$
\begin{aligned}
&\left[x_-(\mathfrak{K}^2 + x_-^2 - x_+^2 - 3y_-^2 + y_+^2) + 2x_+y_-y_+\right]\sin x_-\cosh y_-\\
&-\left[x_+(\mathfrak{K}^2 - x_-^2 + x_+^2 + y_-^2 - 3y_+^2) + 2x_-y_-y_+\right]\sin x_+\cosh y_+\\
&-\left[y_-(\mathfrak{K}^2 + 3x_-^2 - x_+^2 - y_-^2 + y_+^2) - 2x_-x_+y_+\right]\cos x_-\sinh y_-\\
&+\left[y_+(\mathfrak{K}^2 - x_-^2 + 3x_+^2 + y_-^2 - y_+^2) - 2x_-x_+y_-\right]\cos x_+\sinh y_+ = 2\mathfrak{K}(x_-^2 - x_+^2 - y_-^2 + y_+^2)\sin\mathfrak{K},
\end{aligned}
$$

$$
\begin{aligned}
&\left[y_-(\mathfrak{K}^2 + 3x_-^2 - x_+^2 - y_-^2 + y_+^2) - 2x_-x_+y_+\right]\sin x_-\cosh y_-\\
&-\left[y_+(\mathfrak{K}^2 - x_-^2 + 3x_+^2 + y_-^2 - y_+^2) - 2x_-x_+y_-\right]\sin x_+\cosh y_+\\
&+\left[x_-(\mathfrak{K}^2 + x_-^2 - x_+^2 - 3y_-2 + y_+^2) + 2x_+y_-y_+\right]\cos x_-\sinh y_-\\
&-\left[x_+(\mathfrak{K}^2 - x_-^2 + x_+^2 + y_-^2 - 3y_+^2) + 2x_-y_-y_+\right]\cos x_+\sinh y_+ = 4\mathfrak{K}(x_-y_- - x_+y_+)\sin\mathfrak{K}.
\end{aligned}
$$

$\mathcal{PT}$-symmetric solutions: $\mathfrak{n}_2^* = \mathfrak{n}_1 =: \eta + i\kappa$

$$x_+ = \mathfrak{K}\,\eta, \quad y_+ = x_- = 0, \quad y_- = \mathfrak{K}\,\kappa$$

$\mathfrak{K}$

$\mathcal{PT}$-symmetric solutions: $\mathfrak{n}_2^* = \mathfrak{n}_1 =: \eta + i\kappa$

$$x_+ = \mathfrak{K}\,\eta, \quad y_+ = x_- = 0, \quad y_- = \mathfrak{K}\,\kappa$$

$$\left(\frac{\eta^2}{\eta^2+\kappa^2}\right)\cos(\mathfrak{K}\eta) + \left(\frac{\kappa^2}{\eta^2+\kappa^2}\right)\cosh(\mathfrak{K}\kappa) = \cos\mathfrak{K},$$

$$\frac{1}{2}\left[\left(1+\frac{1}{\eta^2+\kappa^2}\right)\eta\,\sin(\mathfrak{K}\eta) - \left(1-\frac{1}{\eta^2+\kappa^2}\right)\kappa\,\sinh(\mathfrak{K}\kappa)\right] = \sin\mathfrak{K}.$$

$\mathfrak{K}$

$\mathcal{PT}$-symmetric solutions: $\mathfrak{n}_2^* = \mathfrak{n}_1 =: \eta + i\kappa$

$$x_+ = \mathfrak{K}\,\eta, \quad y_+ = x_- = 0, \quad y_- = \mathfrak{K}\,\kappa$$

$$\left(\frac{\eta^2}{\eta^2+\kappa^2}\right)\cos(\mathfrak{K}\eta) + \left(\frac{\kappa^2}{\eta^2+\kappa^2}\right)\cosh(\mathfrak{K}\kappa) = \cos\mathfrak{K},$$

$$\frac{1}{2}\left[\left(1+\frac{1}{\eta^2+\kappa^2}\right)\eta\,\sin(\mathfrak{K}\eta) - \left(1-\frac{1}{\eta^2+\kappa^2}\right)\kappa\,\sinh(\mathfrak{K}\kappa)\right] = \sin\mathfrak{K}.$$

$$\mathfrak{n}_1 = \mathfrak{n}_2^* = 3.4 - 0.00342163i$$

$$\mathfrak{K} = 2000.147552$$

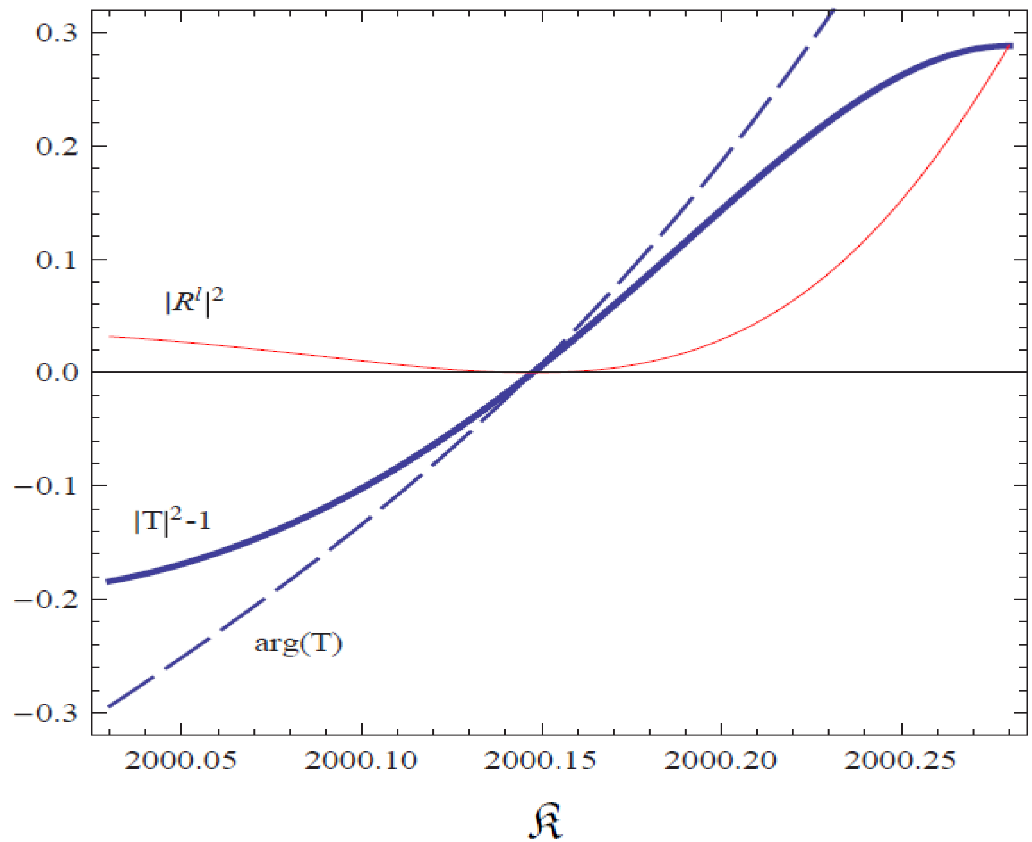
$$L = 300\,\mu\text{m}$$

$$\lambda = 942.408269 \text{ nm}$$

$$|R^l|^2 < 10^{-10}$$

$$|T|^2 - 1 < 10^{-5}$$

$$|R^r|^2 > 0.89$$



Non-$\mathcal{PT}$-symmetric solutions:

$$(x_-^2 - y_-^2)\cos x_- \cosh y_- - (x_+^2 - y_+^2)\cos x_+ \cosh y_+$$
$$+2x_-y_- \sin x_- \sinh y_- - 2x_+y_+ \sin x_+ \sinh y_+ = (x_-^2 - x_+^2 - y_-^2 + y_+^2)\cos \mathfrak{K},$$

$$2x_-y_- \cos x_- \cosh y_- - 2x_+y_+ \cos x_+ \cosh y_+$$
$$-(x_-^2 - y_-^2)\sin x_- \sinh y_- + (x_+^2 - y_+^2)\sin x_+ \sinh y_+ = 2(x_-y_- - x_+y_+)\cos \mathfrak{K},$$

$$\left[x_-(\mathfrak{K}^2 + x_-^2 - x_+^2 - 3y_-^2 + y_+^2) + 2x_+y_-y_+\right]\sin x_- \cosh y_-$$
$$-\left[x_+(\mathfrak{K}^2 - x_-^2 + x_+^2 + y_-^2 - 3y_+^2) + 2x_-y_-y_+\right]\sin x_+ \cosh y_+$$
$$-\left[y_-(\mathfrak{K}^2 + 3x_-^2 - x_+^2 - y_-^2 + y_+^2) - 2x_-x_+y_+\right]\cos x_- \sinh y_-$$
$$+\left[y_+(\mathfrak{K}^2 - x_-^2 + 3x_+^2 + y_-^2 - y_+^2) - 2x_-x_+y_-\right]\cos x_+ \sinh y_+ = 2\mathfrak{K}(x_-^2 - x_+^2 - y_-^2 + y_+^2)\sin \mathfrak{K},$$

$$\left[y_-(\mathfrak{K}^2 + 3x_-^2 - x_+^2 - y_-^2 + y_+^2) - 2x_-x_+y_+\right]\sin x_- \cosh y_-$$
$$-\left[y_+(\mathfrak{K}^2 - x_-^2 + 3x_+^2 + y_-^2 - y_+^2) - 2x_-x_+y_-\right]\sin x_+ \cosh y_+$$
$$+\left[x_-(\mathfrak{K}^2 + x_-^2 - x_+^2 - 3y_-2 + y_+^2) + 2x_+y_-y_+\right]\cos x_- \sinh y_-$$
$$-\left[x_+(\mathfrak{K}^2 - x_-^2 + x_+^2 + y_-^2 - 3y_+^2) + 2x_-y_-y_+\right]\cos x_+ \sinh y_+ = 4\mathfrak{K}(x_-y_- - x_+y_+)\sin \mathfrak{K}.$$

Non-$\mathcal{PT}$-symmetric solutions:

$$(x_-^2 - y_-^2)\cos x_- \cosh y_- - (x_+^2 - y_+^2)\cos x_+ \cosh y_+$$
$$+2x_-y_- \sin x_- \sinh y_- - 2x_+y_+ \sin x_+ \sinh y_+ = (x_-^2 - x_+^2 - y_-^2 + y_+^2)\cos \mathfrak{K},$$

$$2x_-y_- \cos x_- \cosh y_- - 2x_+y_+ \cos x_+ \cosh y_+$$
$$-(x_-^2 - y_-^2)\sin x_- \sinh y_- + (x_+^2 - y_+^2)\sin x_+ \sinh y_+ = 2(x_-y_- - x_+y_+)\cos \mathfrak{K},$$

$$\left[x_-(\mathfrak{K}^2 + x_-^2 - x_+^2 - 3y_-^2 + y_+^2) + 2x_+y_-y_+\right]\sin x_- \cosh y_-$$
$$-\left[x_+(\mathfrak{K}^2 - x_-^2 + x_+^2 + y_-^2 - 3y_+^2) + 2x_-y_-y_+\right]\sin x_+ \cosh y_+$$
$$-\left[y_-(\mathfrak{K}^2 + 3x_-^2 - x_+^2 - y_-^2 + y_+^2) - 2x_-x_+y_+\right]\cos x_- \sinh y_-$$
$$+\left[y_+(\mathfrak{K}^2 - x_-^2 + 3x_+^2 + y_-^2 - y_+^2) - 2x_-x_+y_-\right]\cos x_+ \sinh y_+ = 2\mathfrak{K}(x_-^2 - x_+^2 - y_-^2 + y_+^2)\sin \mathfrak{K},$$

$$\left[y_-(\mathfrak{K}^2 + 3x_-^2 - x_+^2 - y_-^2 + y_+^2) - 2x_-x_+y_+\right]\sin x_- \cosh y_-$$
$$-\left[y_+(\mathfrak{K}^2 - x_-^2 + 3x_+^2 + y_-^2 - y_+^2) - 2x_-x_+y_-\right]\sin x_+ \cosh y_+$$
$$+\left[x_-(\mathfrak{K}^2 + x_-^2 - x_+^2 - 3y_-2 + y_+^2) + 2x_+y_-y_+\right]\cos x_- \sinh y_-$$
$$-\left[x_+(\mathfrak{K}^2 - x_-^2 + x_+^2 + y_-^2 - 3y_+^2) + 2x_-y_-y_+\right]\cos x_+ \sinh y_+ = 4\mathfrak{K}(x_-y_- - x_+y_+)\sin \mathfrak{K}.$$

- $|x_\pm| = |\mathrm{Re}(\mathfrak{K}\mathfrak{n}_\pm)|/2$ and $\mathfrak{K}$ are typically large numbers.

New variables: $x_\pm = 2\pi m_\pm + \dfrac{\gamma_\pm}{2\pi m_\pm}$, $\mathfrak{K} = 2\pi m_0 + \dfrac{\gamma_0}{2\pi m_0}$

$$m_+, m_0 \in \mathbb{Z}^+,\ m_- \in \mathbb{Z},\ \gamma_\pm, \gamma_0 \in \mathbb{R}$$

A Non-$\mathcal{PT}$-symmetric left-invisible sample:

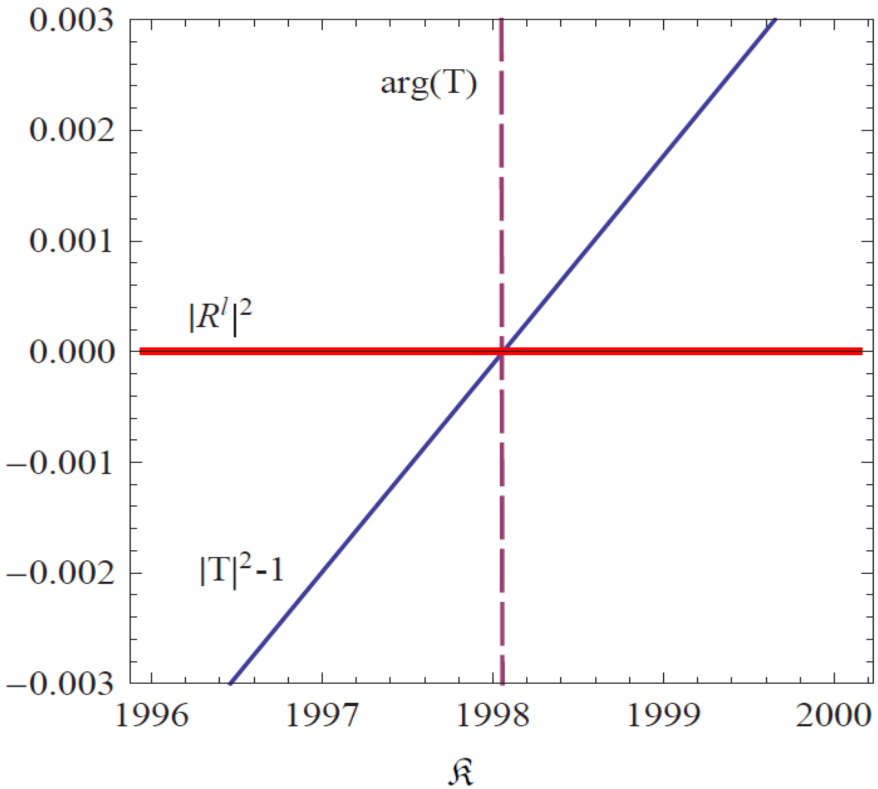
$$\mathfrak{n}_1 = 3.402510 + i(6.062508 \times 10^{-4})$$
$$\mathfrak{n}_2 = 1.402514 - i(1.788281 \times 10^{-3})$$
$$\mathfrak{K} = 1998.049925$$
$$L = 300\ \mu\text{m}$$
$$\lambda = 943.397644\ \text{nm}$$

$\big||T|^2 - 1\big| < 2.1 \times 10^{-5}$

$|\arg(T)| < 3.2 \times 10^{-3}$

$|R^l|^2 < 2.8 \times 10^{-6}$

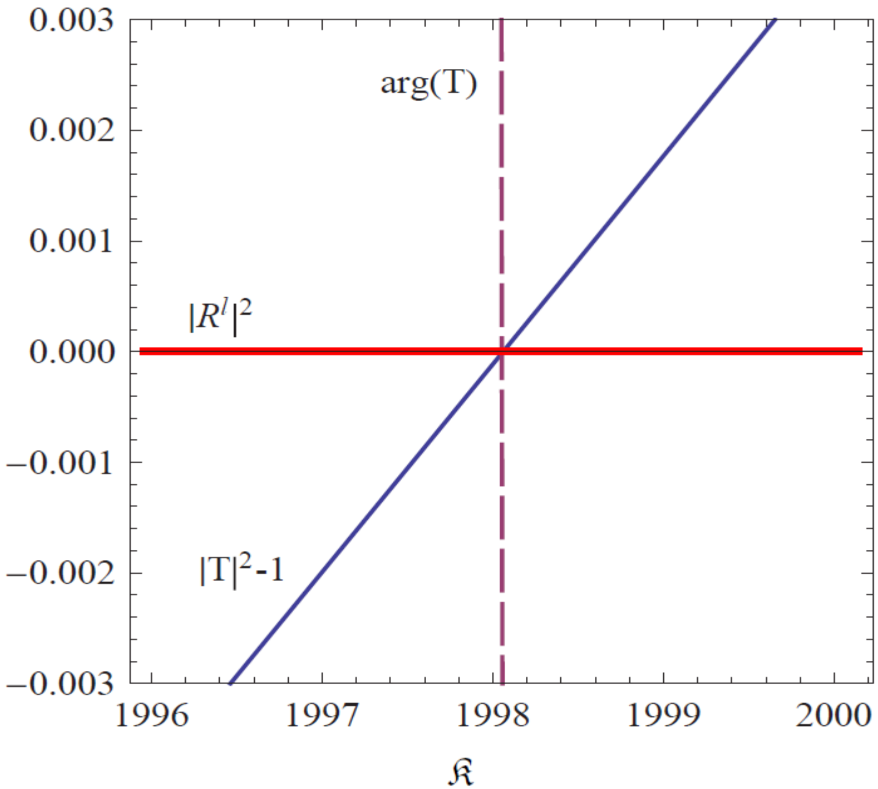
$|R^r|^2 > 14.1$

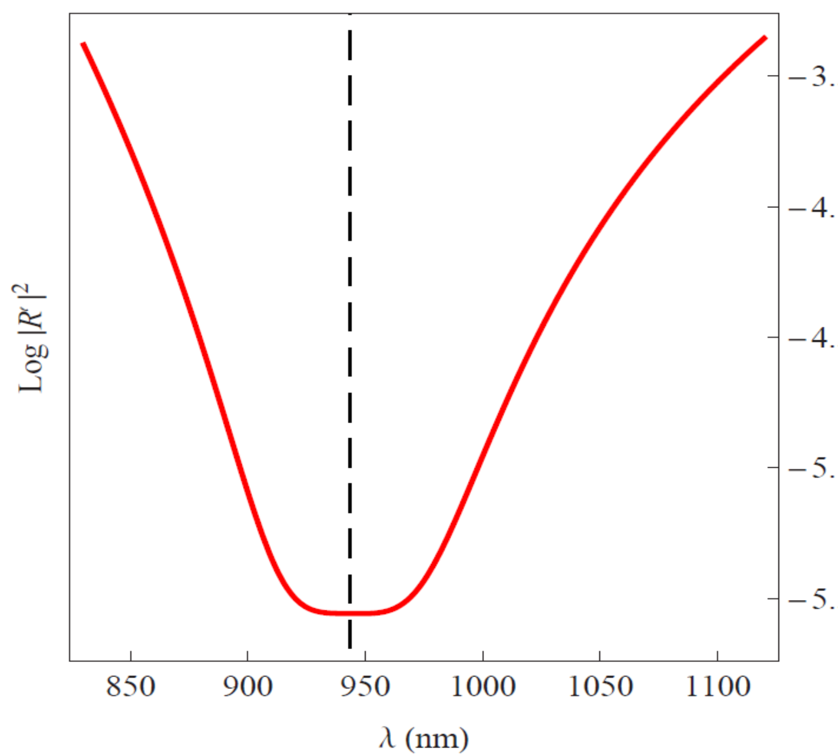


A Non-$\mathcal{PT}$-symmetric left-invisible sample:

$$\mathfrak{n}_1 = 3.402510 + i(6.062508 \times 10^{-4})$$
$$\mathfrak{n}_2 = 1.402514 - i(1.788281 \times 10^{-3})$$
$$\mathfrak{K} = 1998.049925$$
$$L = 300\ \mu\text{m}$$
$$\lambda = 943.397644\ \text{nm}$$

$$\big||T|^2 - 1\big| < 2.1 \times 10^{-5}$$
$$|\arg(T)| < 3.2 \times 10^{-3}$$
$$|R^l|^2 < 2.8 \times 10^{-6}$$
$$|R^r|^2 > 14.1$$





Conclusion

Formal question of how one can define a unitary quantum system using an apparently **non-Hermitian Hamiltonian operators with a real spectrum** led to the idea of **changing the inner product** of the Hilbert space and methods for **constructing metric operators**.

Conclusion

Formal question of how one can define a unitary quantum system using an apparently **non-Hermitian Hamiltonian operators with a real spectrum** led to the idea of **changing the inner product** of the Hilbert space and methods for **constructing metric operators**.

These fail when the point spectrum involves a defective eigenvalue, that is when an **exceptional point** arises, or the continuous spectrum includes a **spectral singularity**.

Conclusion

Formal question of how one can define a unitary quantum system using an apparently **non-Hermitian Hamiltonian operators with a real spectrum** led to the idea of **changing the inner product** of the Hilbert space and methods for **constructing metric operators**.

These fail when the point spectrum involves a defective eigenvalue, that is when an **exceptional point** arises, or the continuous spectrum includes a **spectral singularity**.

Physical implications of exceptional points are well-studied for more than two decades. For example they lead to **geometric phases** that have been explored experimentally. The physical meaning of a spectral singularity has been understood more recently. Spectral singularities can be identified with **zero-width resonances**. In optics they correspond to **lasing at threshold gain**. Their time-reversal gives rise to **antilasing**.

The phenomenon of **self-dual spectral singularity** is both $\mathcal{P}$- and $\mathcal{T}$- and hence $\mathcal{PT}$-symmetric in nature.

This does not however mean that only $\mathcal{PT}$-symmetric potentials are capable of supporting spectral singularities. There are indeed **non-$\mathcal{PT}$-symmetric** potentials/optical systems that have self-dual spectral singularities/function as **CPS-lasers** and are easier to realize in practice.

The phenomenon of **self-dual spectral singularity** is both $\mathcal{P}$- and $\mathcal{T}$- and hence $\mathcal{PT}$-symmetric in nature.

This does not however mean that only $\mathcal{PT}$-symmetric potentials are capable of supporting spectral singularities. There are indeed **non-$\mathcal{PT}$-symmetric** potentials/optical systems that have self-dual spectral singularities/function as **CPS-lasers** and are easier to realize in practice.

Unidirectional reflectionlessness and invisibility are not $\mathcal{P}$- and $\mathcal{T}$-symmetric, but $\mathcal{PT}$**-symmetric** in nature.

As far as I know, this is the only physical phenomenon where $\mathcal{PT}$**-symmetry plays is basic role**.

References:

Pseudo-Hermitian QM:
- IJGMMP **7**, 1191 (2010); arXiv:0810.5643

Physical aspects of spectral singularities:
- PRL **102**, 220402 (2009); arXiv:0901.4472
- PRA **80**, 032711 (2009); arXiv:0908.1713
- PRA **83**, 045801 (2011); arXiv:1102.4695

Semiclassical & perturbative evaluation of spec. singularities:
- PRA **84**, 023809 (2011); arXiv:1105.4462
- (with Rostamzadeh) PRA **86**, 022103 (2012); arXiv:1204.2701

Spectral singularities in spherical medium (with Sarısaman):
- PLA **375**, 3387 (2011); arXiv:1107.1873
- Proc. R. Soc. A **468**, 3224 (2012); arXiv:1205.5472

Self-dual spectral singularities & unidir. invisibility:
- JPA **45**, 444024 (2012); arXiv:1205.4560
- PRA **87**, 012103 (2013); arXiv:1206.0116

Nonlinear extensions: arXiv:1303.2501 & 1303.4874.

References:

Pseudo-Hermitian QM:

- IJGMMP **7**, 1191 (2010); arXiv:0810.5643

Physical aspects of spectral singularities:

- PRL **102**, 220402 (2009); arXiv:0901.4472
- PRA **80**, 032711 (2009); arXiv:0908.1713
- PRA **83**, 045801 (2011); arXiv:1102.4695

Semiclassical & perturbative evaluation of spec. singularities:

- PRA **84**, 023809 (2011); arXiv:1105.4462
- (with Rostamzadeh) PRA **86**, 022103 (2012); arXiv:1204.2701

Spectral singularities in spherical medium (with Sarısaman):

- PLA **375**, 3387 (2011); arXiv:1107.1873
- Proc. R. Soc. A **468**, 3224 (2012); arXiv:1205.5472

Self-dual spectral singularities & unidir. invisibility:

- JPA **45**, 444024 (2012); arXiv:1205.4560
- PRA **87**, 012103 (2013); arXiv:1206.0116

Nonlinear extensions: arXiv:1303.2501 & 1303.4874.

Thank you for your attention.