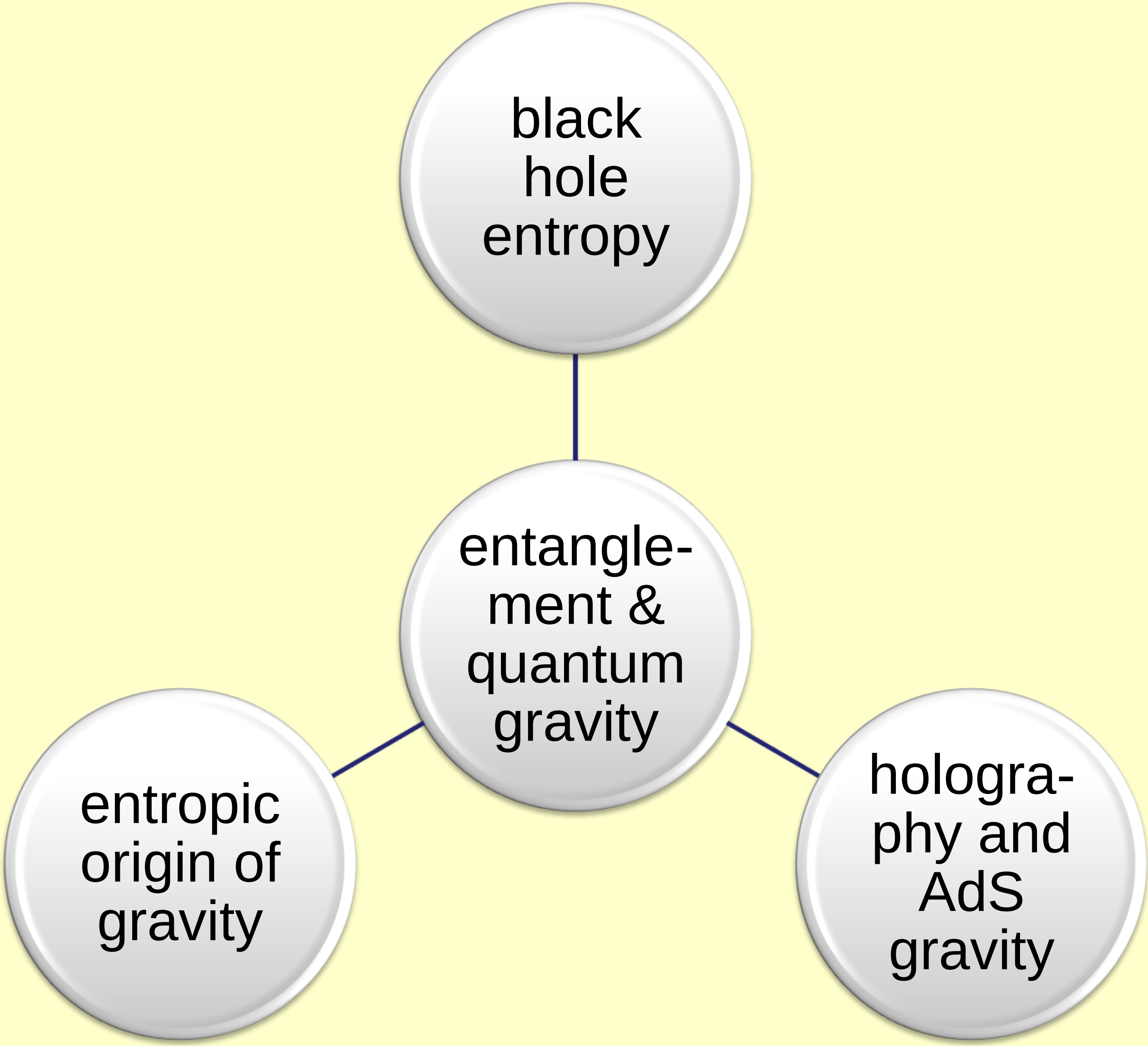
Lectures on Entanglement Entropy:

an introduction to QFT computations and related aspects of quantum gravity

Lecture 2

Dmitri V. Fursaev
Dubna University &
Bogoliubov Laboratory
JINR

25.04.2013
IPM, Tehran


black hole entropy
entangle-ment & quantum gravity
entropic origin of gravity
hologra-phy and AdS gravity

Plan of lectures

Lecture 1:

-Entanglement entropy (definitions and basic properties)
- EE and Renyi entropy (REE): methods of computations in free QFT's (spectral geometry and etc)

Lecture 2:

-Holographic EE (HEE)
- motivations for HEE
- HEE: how it works
- HEE and conformal anomalies

Lecture 3:

- Bekenstein-Hawking entropy of black holes
- Entanglement in Emergent gravity
- Entropic gravity

entanglement has to do with quantum gravity:

● possible source of the entropy of a black hole (states inside and outside the horizon);

● $d=4$ supersymmetric BH's are equivalent to 2, 3,… qubit systems

● entanglement entropy allows a *holographic interpretation* for CFT's with AdS duals

Toward a holographic description of Entanglement Entropy in CFT's

‘Holography in a nutshell’

$$I_{gravity,D+1} = \frac{1}{16\pi G_{D+1}} \int \sqrt{g}\, d^{D+1}x \,(R + 2\Lambda) + \text{b.t.}$$

$$I_{gravity,D+1} = F_D(T)/T$$

$F_D(T)$ – free energy of a CFT

$$F_{D=2}(T) \sim cT^2 L, \quad c = \frac{3l}{2G}, \quad \Lambda \sim \frac{1}{l^2}$$

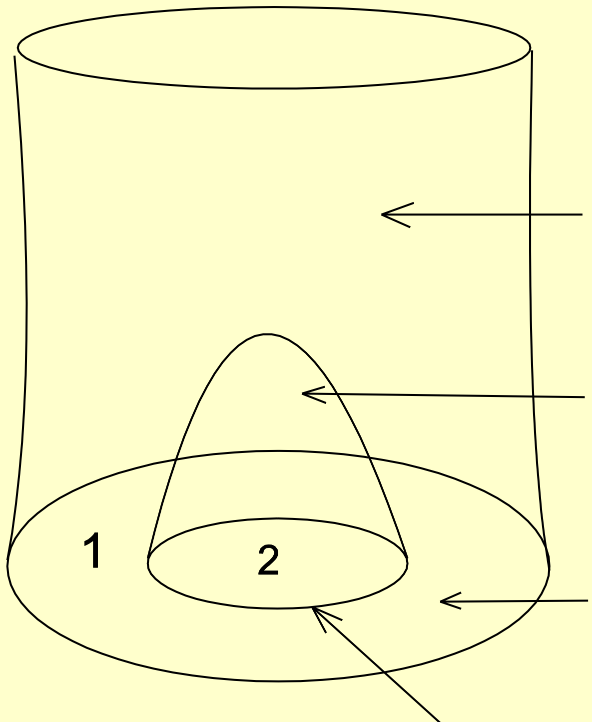
$D = 4$: Type IIB string theory on $AdS_5 \times S_5$

correponds to $\mathbb{N} = 4$ $SU(N)$ SYM :

$\mathrm{g}_{YM}{}^2 \sim g_{string}$, $\quad l \sim \mathrm{g}_{YM}{}^2 N$, N is a 5-form flux on S^5

Holographic Formula for Entanglement Entropy (n=1)

Ryu and Takayanagi,
hep-th/0603001, 0605073



AdS_{d+1} (bulk space)

$\tilde{B}$ minimal (least area) surface in the bulk

4d space-time manifold (asymptotic boundary of AdS)

B separating surface

entropy of entanglement

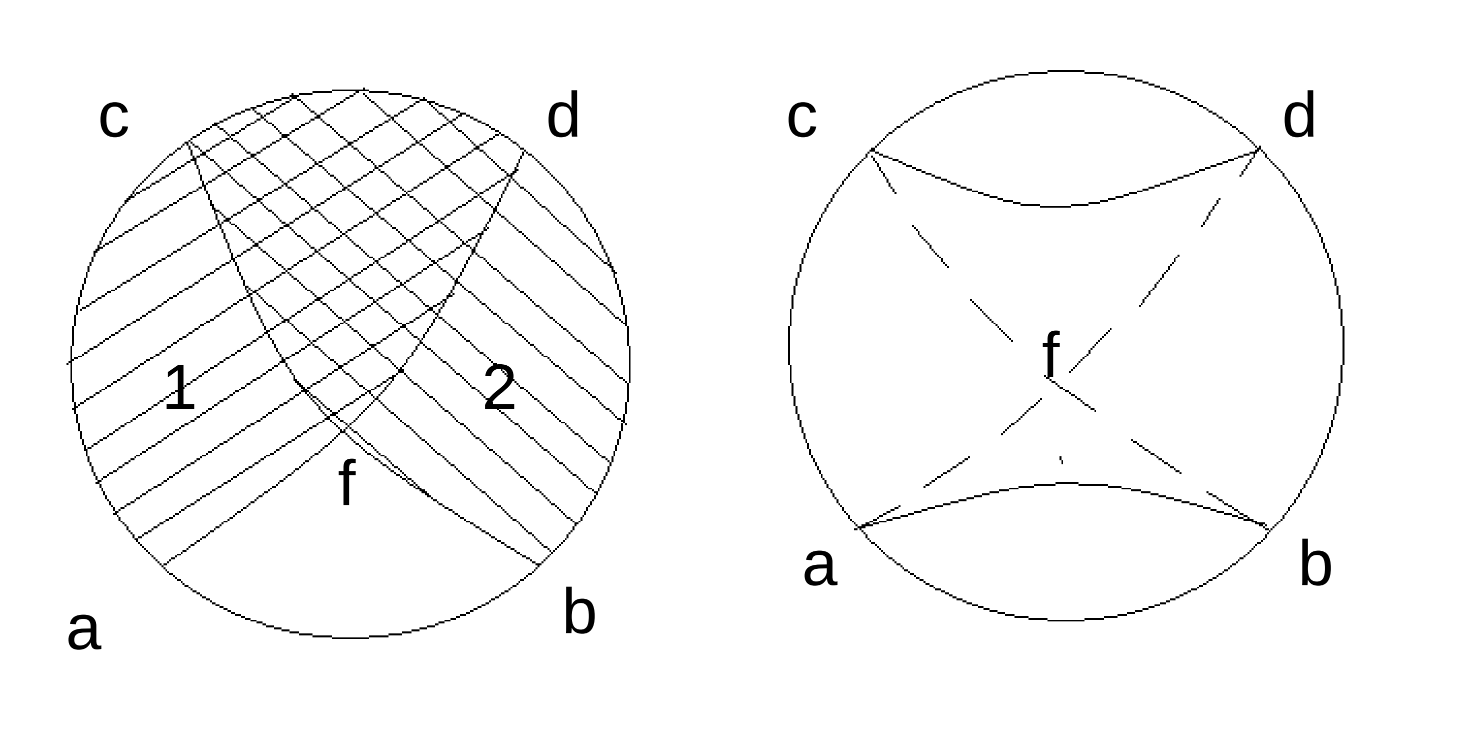
$$\boxed{S = \frac{\tilde{A}}{4G^{(d+1)}}}$$

is measured in terms of the area of $\tilde{B}$

$G^{(d+1)}$ is the gravity coupling in AdS

The holographic formula at work

strong subadditivity: $S_1 + S_2 \geq S_{1\cup 2} + S_{1\cap 2}$



$$S_1 = A_{ad} \ , \quad S_2 = A_{bc}$$

$$S_1 + S_2 = A_{ad} + A_{bc} = A_{af} + A_{fd} + A_{bf} + A_{fc} =$$

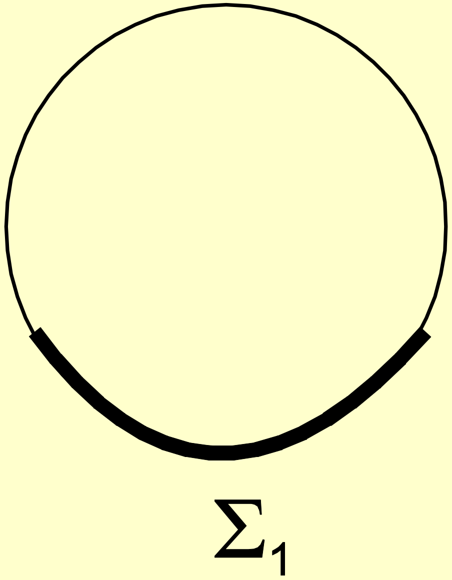
$$(A_{af} + A_{bf}) + (A_{fd} + A_{fc}) \geq A_{ab} + A_{dc} = S_{1\cup 2} + S_{1\cap 2}$$

entanglement in 2D CFT

L_1 is the length of Σ_1

$$S = \frac{c}{3} \ln\left(\frac{L}{\pi a} \sin \frac{\pi L_1}{L} \right)$$

ground state entanglement for a system on a circle



c – is a central charge

example in d=2: CFT on a circle

$$ds^2 = l^2\left(d\rho^2 - \cosh^2\rho dt^2 + \sinh^2\rho d\varphi^2\right)$$

l - AdS radius

$$\Sigma_1 \to \frac{2\pi L_1}{L} \qquad ds^2{}_{CFT} = ds^2\Big|_{\rho=\rho_0}$$

$$\cosh\frac{A}{l} = 1 + 2\sinh^2\rho_0 \sin^2\left(\frac{\pi L_1}{L}\right)$$

A is the length of the geodesic in AdS

$$e^{\rho_0} = \frac{L}{a}$$ - UV cutoff

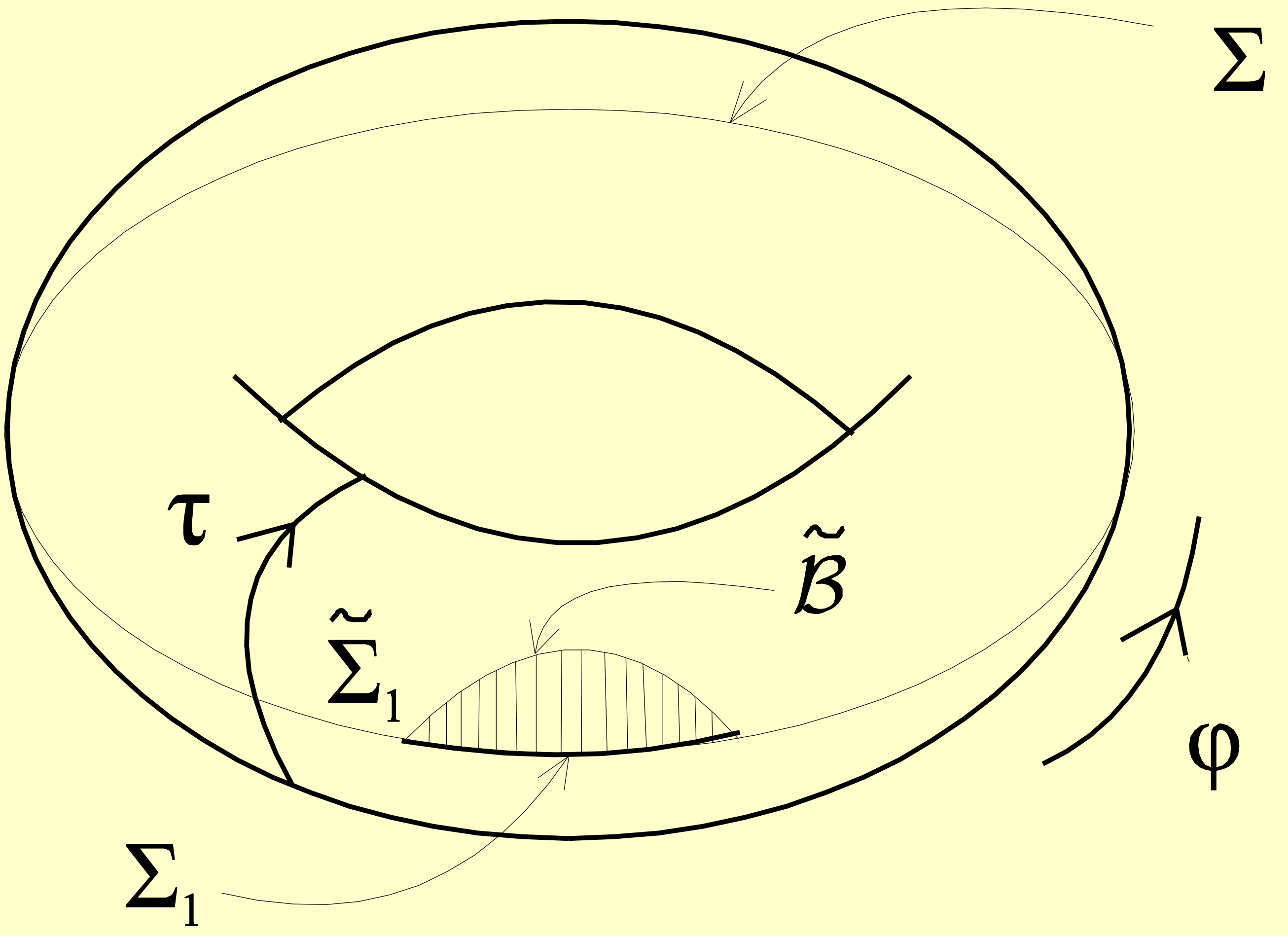
$$\boxed{S = \frac{A}{4G_3} = \frac{c}{3}\ln\left(e^{\rho_0}\sin\left(\frac{\pi L_1}{L}\right)\right)}$$

-holographic formula reproduces the entropy for a ground state entanglement

$$c = \frac{3l}{2G_3}$$ - central charge in d=2 CFT

a CFT on a circle at a finite temperature and BTZ black hole

Euclidean BTZ black hole



The holographic formula at work: higher dimensions

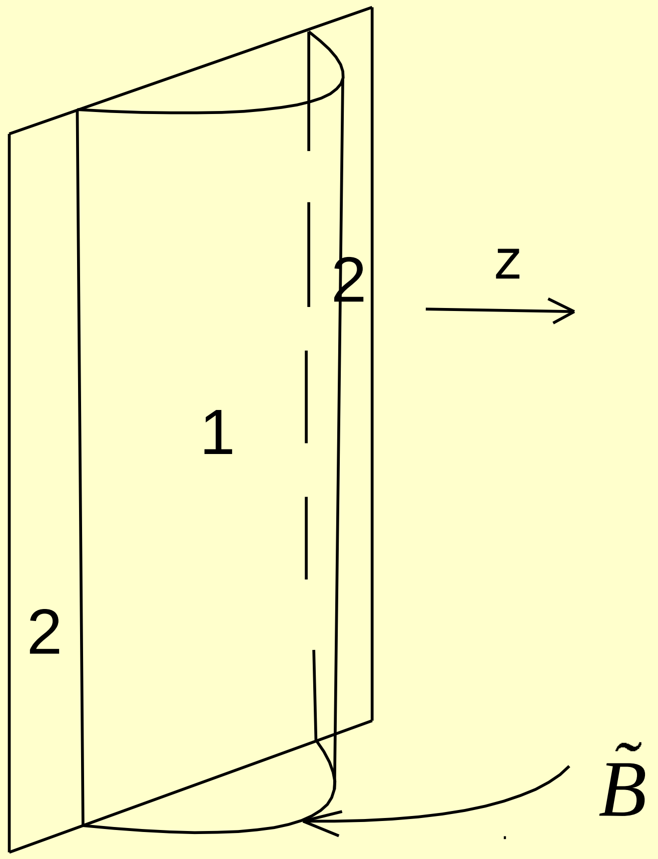
$$S = \frac{A(\tilde{B})}{4G_N^{(D+1)}}$$

$$A(\tilde{B}) - \text{volume of } \tilde{B};$$

Consider the entanglement entropy for N=4 super Yang-Mills in d=4

$$AdS_5$$

a simple example: entanglement of a strip



$$ds_5^2 = \frac{l^2}{z^2}(dz^2 + ds_4^2)$$

$$\tilde{A} \sim \frac{l^3}{a^2} A$$

a – is IR cutoff

$$S = \frac{\tilde{A}}{4G_5} \sim \frac{l^3}{a^2 G_5} A \sim \frac{N^2}{a^2} A$$

$$\frac{l^3}{G_5} \sim N^2, \qquad (SU(N))$$

the Plateau Problem

(Joseph Plateau, 1801-1883)

It is a problem of finding a least area surface (minimal surface) for a given boundary

soap films:

$$k = h(p_1 - p_2)$$ - equilibrium equation

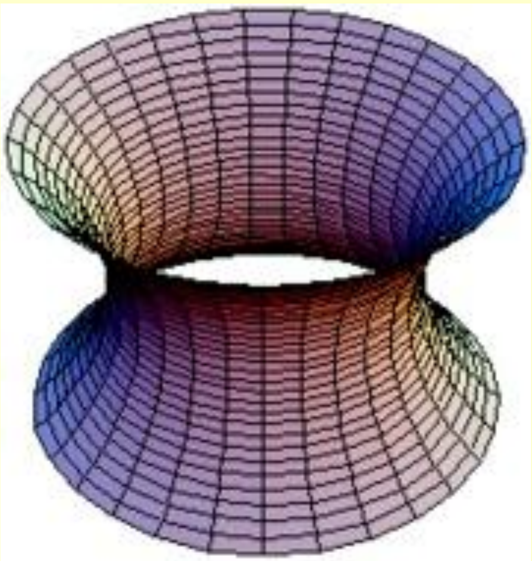
k - the mean curvature

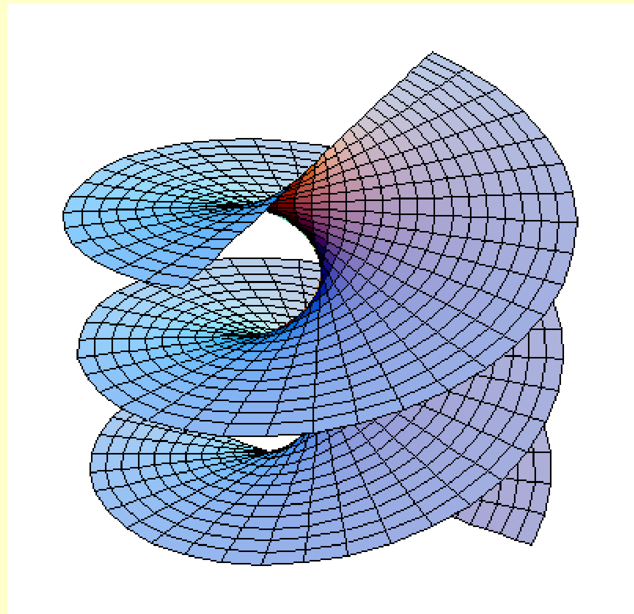
h^{-1} - surface tension

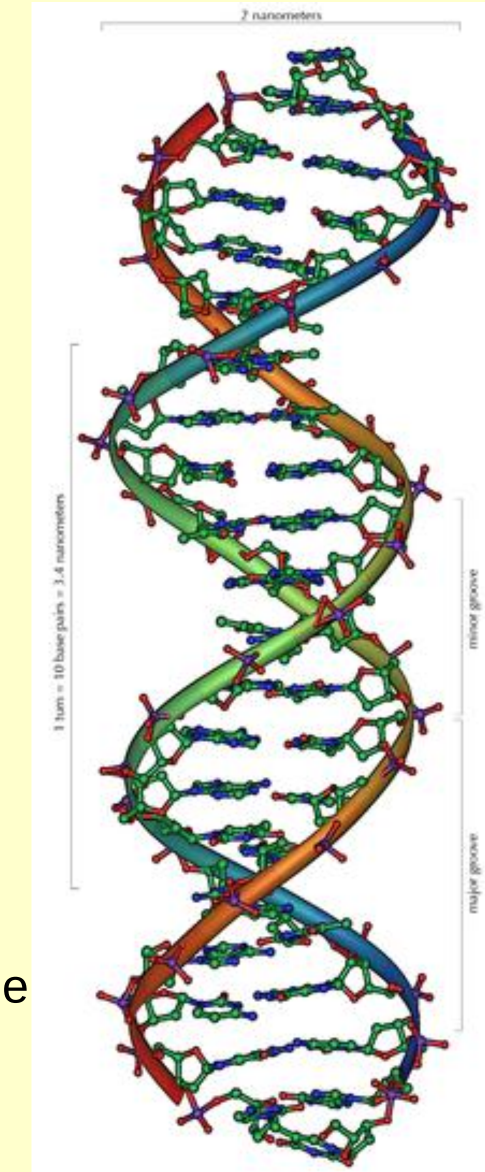
$p_1 - p_2$ -pressure difference across the film

the Plateau Problem

simple surfaces







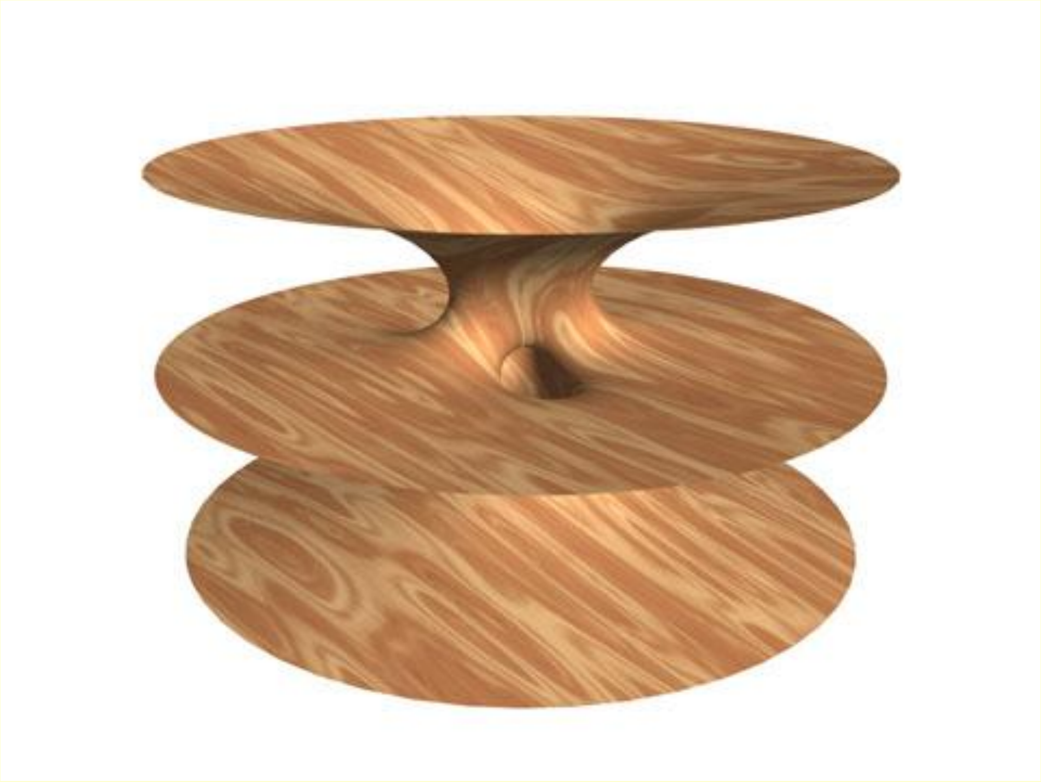
catenoid is a three-dimensional shape made by rotating a catenary curve (discovered by L.Euler in 1744)

helicoid is a ruled surface, meaning that it is a trace of a line

The structure of part of a DNA is double helix

the Plateau Problem

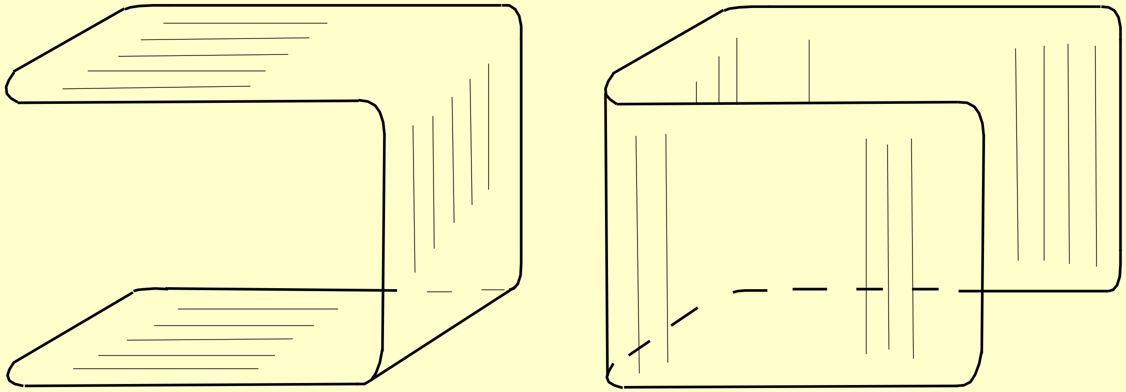
other embedded surfaces (without self intersections)



Costa’s surface (1982)

the Plateau Problem

there are no unique solutions in general (especially for non-trivial boundaries!)



A B

Asymptotics at AdS

$\tilde{B}-$ is a holographic surface in the bulk;

$\partial\tilde{B}-$ belongs to conformal class of B;

$\tilde{M}$ – asymptotically AdS solution to:

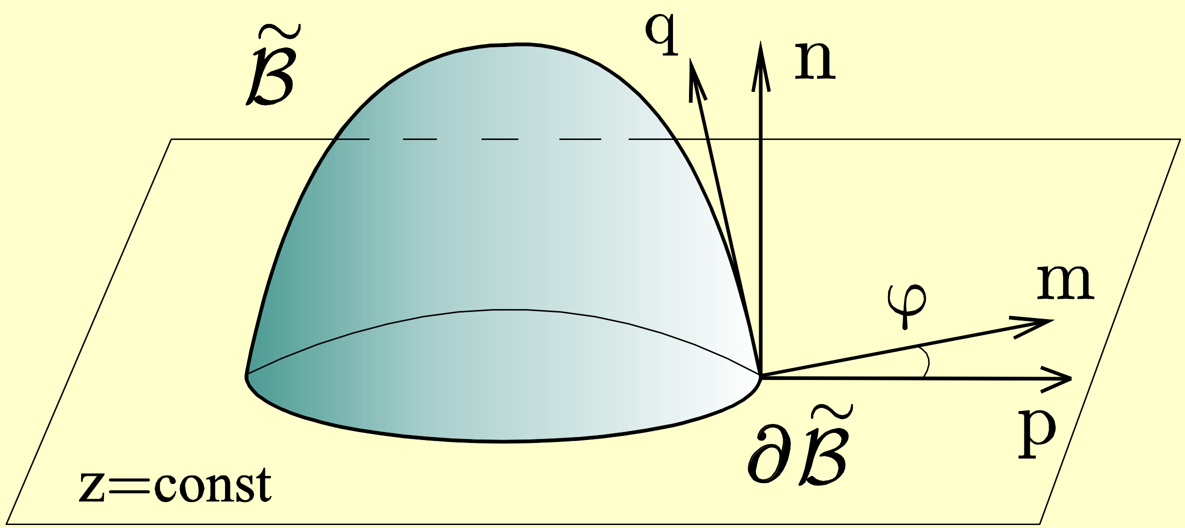
$$\tilde{R}_{MN} - \frac{1}{2}\tilde{R}\,\tilde{g}_{MN} - \frac{3}{l^2}\tilde{g}_{MN} = 0$$

$$ds^2 = \tilde{g}_{MN}dx^M dx^N = z^{-2}(dz^2 + g_{\mu\nu}(z,x)dx^\mu dx^\nu),$$

Fefferman-Graham (FG) expansion in $d = 4$

$$g_{\mu\nu}(z,x) = g_{\mu\nu}(x) - \frac{z^2}{2}\left(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\right) + \ldots$$

FG expansion for the area of a minimal surface



Let $\tilde{B}$ – be a minimal codimension 2 hypersurface in $\tilde{M}$, ($\partial\tilde{B}$ conformal to B)

one needs to find an analog of FG expansion for the metric on $\tilde{B}$

$$ds^2(\tilde{B}) = z^{-2}\left(\frac{dz^2}{\cos^2\varphi} + \sigma_{ab}(z,y)dy^a dy^b\right)$$

FG expansion (continued)

tilt angle: $\varphi = \frac{z}{2} k + \ldots, \quad k$ $-$ extrinsic curvature of B

$x^\mu = x^\mu(z, y), \quad \mu \neq \tau, \quad \tau = const$ $-$ embedding equations of $\tilde{B}$

$x^\mu = x^\mu(y), \quad z = const$ $-$ embedding equations of B

$$x^\mu(z, y) = x^\mu(y) + x^\mu{}_{(1)}(y) z + x^\mu{}_{(2)}(y) z^2 + \ldots$$

$$x^\mu(z, y) = x^\mu(y) \boxed{- \frac{k}{2(d-2)} \bar{p}^\mu z^2 + \ldots,} \; \mu \neq \tau,$$

$$\bar{p}^\mu = z p^\mu, \quad k_{ab} = x^\mu_{,a} x^\nu_{,b} \bar{p}_{\mu;\nu} \; ;$$

$$\sigma_{ab}(z, y) = \sigma_{ab}(y) - \frac{z^2}{2} \left(k k_{ab} + R_{ab} - \frac{1}{6} \sigma_{ab} R \right) + \ldots, \quad R_{ab} = R_{\mu\nu} x^\mu_{,a} x^\nu_{,b},$$

Holographic entanglement entropy

volume of $\tilde{B}$: $\quad A(\tilde{B}) = \frac{1}{2\varepsilon^2} A(B) + \frac{\pi}{2}(F_a + F_c + F_b) \ln \frac{\mu}{\varepsilon} + \ldots$

$z = \varepsilon -$ position of the boundary (a UV cutoff in CFT)

$$\boxed{S(B) = \frac{A(\tilde{B})}{4G_5} \sim \frac{N^2 \Lambda^2}{4\pi} A(B) + \frac{1}{4} N^2 (F_a + F_c + F_b) \ln \mu\Lambda + \ldots}$$

use AdS/CFT dictionary: $\frac{1}{G_5} = \frac{2N^2}{\pi}, \quad \varepsilon = 1/\Lambda$

one reproduces correctly the structutre of the leading divergences and exact value

of the logarithmic part of the entropy

‘derivation’ of holographic entanglement entropy:

D.F. JHEP 0609 (2006) 018, e-Print: hep-th/0606184

an attempt to find more arguments

Why ‘derivation’ of Ryu-Takayanagi formula is important?

- practical issues like its modifications by quantum corrections and etc;
- fundamental issues like understanding entanglement entropy
in quantum gravity

the idea of origin of the holographic formula

$$\boxed{Z_{CFT} = Z_{AdS}}$$

$$Z_{AdS}(M_n) = \int_{\tilde{M}_n :\, \partial\tilde{M}_n = M_n} [Dg] e^{-W[g]}, \quad \text{– partition function for Renyi entropy of ordrer } n$$

$\ln Z_{AdS}(M_n) \simeq -W(n)$ – action at a stationary point, the holographic entropy is

$$S_{AdS}(n) \equiv \frac{1}{1-n}\left(\ln Z_{AdS}(M_n) - n \ln Z_{AdS}(M_{n=1})\right) \simeq \frac{1}{n-1}\left(W(n) - nW(1)\right)$$

taking a naive limit ($n \to 1$) one has (assuming bulk spaces have conical singularities)

$$W[M_n] \to I[M_n] = -\frac{1}{16\pi G_{d+1}} \int_{\tilde{M}_n/\tilde{B}} R\sqrt{g}\, d^{d+1}x + \frac{1}{4G_{d+1}}(n-1)A[\tilde{B}],$$

$$S_{AdS}(n \to 1) \simeq \frac{1}{4G_{d+1}} A[\tilde{B}] = S_{CFT}$$

saddle point approximation requires $A[\tilde{B}]$ to be a minimal hypersurface!

problems with these arguments:

1. The derivation does not reproduce the entanglement Renyi entropy (M. Headrick, Phys. Rev. D82 (2010) 126010, arXiv: 1006.0047[hep-th]);

2. Bulk manifolds with conical singularities cannot be stationary configurations of the AdS partition function (M. Headrick,…);

3. The behavior of string theory on singular manifolds is not known;

4. Path integral integral representation of the Renyi entropy for non-integer index (n -> 1) requires certain conditions which are not always satisfied (H.Cassini, M.Huerta, arXiv: 1203.4007[hep-th]);

5. …..

We try to understand better the origin of RT formula by studying holographic entanglement Renyi entropy (ERE)

Lecture I:

- in general, $\hat{\rho}^{\alpha}$ cannot be represented as an analogous evolution operator (for non-integer α),
- there may not exist a geometrical construction of a background manifold with conical singularities which corresponds to $\mathrm{Tr}\,\hat{\rho}^{\alpha}$
- modular Hamiltonian is a non-local operator

we do not assume path integral representation for non-integer α,

or corresponding background geometries

we use the following method: do computations for integer α and

consider an analytical continuation (see, e.g. S.N. Solodukhin and

D. Nesterov, arXiv:1007.1246, NP B842 (2011) 141)

holographic representation of ERE may exist for integer indexes since ERE is represented in terms of partition functions and there is no problem with path integral representation of the reduced density matrix

$$S^{(n)}(T)=\frac{\ln Z(n,T)-n\ln Z(T)}{1-n}$$

We use a bottom-up approach to holographic ERE :
by trying reconstruct bulk quantities from boundary ones

conditions:

-holographic ERE is a functional set on a minimal surface;

-bulk geometry where the minimal surface is lying
does not depend on the Renyi index (at least locally) – same property as for the boundary geometry;

Holographic Renyi Entropy (a suggestion)

$$S^{(n)}(B)=\frac{1}{4G_N^{(5)}}\Big(f(\gamma_n)A(\tilde{B})+2\pi(\tilde{a}(\gamma_n)\tilde{F}_a+\tilde{c}(\gamma_n)\tilde{F}_c+\tilde{b}(\gamma_n)\tilde{F}_b)\Big)+\ldots.$$

$A(\tilde{B})-$ volume of $\tilde{B}$; $\quad \partial\tilde{B}=B$;

$\tilde{F}_a,\ \tilde{F}_c,\ \tilde{F}_b -$ are some local (bulk) invariant functionals set on $\tilde{B}$;

$f(\gamma_n),\tilde{a}(\gamma_n),\tilde{c}(\gamma_n),\tilde{b}(\gamma_n)-$ some coefficient functions;

to reproduce Ryu-Takayanagi formula for entanglement entropy

$$S^{(n=1)}=\frac{1}{4G_N^{(5)}}A(\tilde{B})$$

$$f(1)=1,\ \ \tilde{a}(1)=\tilde{c}(1)=\tilde{b}(1)=0$$

$\tilde{F}_a, \tilde{F}_c, \tilde{F}_b -$ are fixed by conformal invariance (should not depend on the coupling)

$$\tilde{F}_a \to F_a, \ \tilde{F}_c \to F_c, \quad \tilde{F}_b \to F_b,$$

$$F_a = -\frac{1}{2\pi}\int_B \sqrt{\sigma} d^2 x\, R(B) \quad , \qquad F_c = \frac{1}{2\pi}\int_B \sqrt{\sigma} d^2 x\, C_{\mu\nu\lambda\rho} n_i^\mu n_j^\nu n_i^\lambda n_j^\rho ,$$

$$F_b = \frac{1}{2\pi}\int_B \sqrt{\sigma} d^2 x \left(\frac{1}{2}\mathrm{Tr}(k_i)\mathrm{Tr}(k_i) - \mathrm{Tr}(k_i k_i) \right) ,$$

the strategy is to fix $\tilde{F}_a, \tilde{F}_c, \tilde{F}_b$ in the limit of weak couplings

$f(\gamma_n), \tilde{a}(\gamma_n), \tilde{c}(\gamma_n), \tilde{b}(\gamma_n) -$ may depend on the coupling, extra information is required

Asymptotics at AdS

$$2\tilde{R}_{KLMN}(z,y)l^K m^L l^M m^N = -2 + z^2 C_{ijij}(y) + \ldots,$$

$$\tilde{R}_B(z,y) = -6 + z^2\left(C_{ijij} + \frac{k^2}{3} - \mathrm{Tr}(k^2)\right) + \ldots,$$

$$K_{MN}K^{MN} = -z^2\left(\frac{k^2}{3} - \mathrm{Tr}(k^2)\right) + \ldots,$$

$\tilde{R}_{KLMN}$ – Riemann tensor of $\tilde{M}$, $\tilde{\sigma}$ – metric induced on $\tilde{B}$,

l, m – normal vectors of $\tilde{B}$,

l – is time-like, $(l \cdot m) = 0$, K_{MN} – extrinsic curvature tensor of $\tilde{B}$ for m^N

Asymptotics at AdS

If we put:

$$\tilde{F}_c = \frac{1}{\pi}\int_{\tilde{B}} \sqrt{\tilde{\sigma}}\, d^3y \left[\tilde{R}_{KLMN} l^K m^L l^M m^N + \frac{1}{l^2} \right]$$

$$\tilde{F}_b = -\frac{1}{2\pi}\int_{\tilde{B}} \sqrt{\tilde{\sigma}}\, d^3y\, K_{MN} K^{MN}$$

it follows that:

$$\tilde{F}_{b,c} = F_{b,c}\, l \ln\frac{\mu}{z} + \ldots$$

no other invariants which yield Weyl invariant structures appear:

$\tilde{R}, \tilde{R}_{KL}$ – are constant (gravity eqs.)

the 3d functional, $\tilde{R}_B$, is not independent (Gauss-Codazzi eq.)

derivation of holographic ERE: same approach

$$Z_{CFT} = Z_{AdS}$$

$$Z_{AdS}(M_n) = \int_{\tilde{M}_n :\, \partial\tilde{M}_n = M_n} [Dg]e^{-W[g]}, \quad - \text{partition function for}$$

Renyi entropy of ordrer n

$$\boxed{W[\tilde{M}_n] - ?}$$

quantum effective action is well defined on manifolds with conical singularities, classical action is not, unless it is linear in curvature;

local part of quantum action is a polynomial in index n

bulk gravity action on singular backgrounds:

$W[M_n]$ is an effective (not classical) action,

$$W[M_n] \neq I[M_n]$$

contribution of conical singularities in the effective action at $n > 1$ has a "non-classical" form (experience with one-loop computations)

$$W[M_n] = -\frac{1}{16\pi G_{d+1}} \int_{\tilde{M}_n/\tilde{B}} R\sqrt{g}\, d^{d+1}x + (n-1) S(n, \tilde{B}),$$

$$S(n, \tilde{B}) \neq A(\tilde{B}),$$

$$S_{AdS}(n) \simeq \frac{1}{n-1}\left(W(n) - nW(1)\right) = S(n, \tilde{B});$$

$S_{AdS}(n)$ coincides with ERE if $S(n, \tilde{B})$ is a holographic ERE

more on variation procedure:

strictly speaking for the holographic Renyi entropy one should choose a surface $\tilde{B}_n$ which extremizes the functional $S(n,\tilde{B})$

$$\delta S(n,\tilde{B}_n) = 0$$

$\tilde{B}_n$ is not minimal, but one expects that

$$\tilde{B}_n = \tilde{B} + \text{correction}$$

$$S(n,\tilde{B}_n) = S(n,\tilde{B}) + \left(\text{correction}\right)^2$$

Thus, ERE may be reproduced by some holographic arguments

No problem with the limit $n \to 1$!

singular manifolds in semiclassical approximation of holographic partition function:

boundary conditions imply two relevant sets of manifolds in the bulk

$$Z_{AdS}(M_n) = \int\limits_{\substack{\tilde{M}_n:\partial\tilde{M}_n=M_n \\ \text{singular}}} [Dg]e^{-W[g,n]} + \int\limits_{\substack{\tilde{M}_n:\partial\tilde{M}_n=M_n \\ \text{regular}}} [Dg]e^{-I[g]}$$

since $W[g,n] \neq I[g]$ the 2 sets are separated

(in the Einstein gravity two sets can be joined since the Einstein action

is well-defined in the presence of conical singularities);

therefore, one should look for the saddle point approximation insisde

a class of singular manifolds, then a singular manifold should be an extremum

without matter sources

it is an open question if regular manifolds in the bulk with singular b.c.

may be relevant as well

To summarize:

There is a ‘scenario’ for a holographic formula of entanglement Renyi entropy which allows “n -> 1” limit and yields holographic entanglement entropy

thank you for attention