Lectures on Entanglement Entropy:

an introduction to QFT computations and related aspects of quantum gravity

Lecture 3

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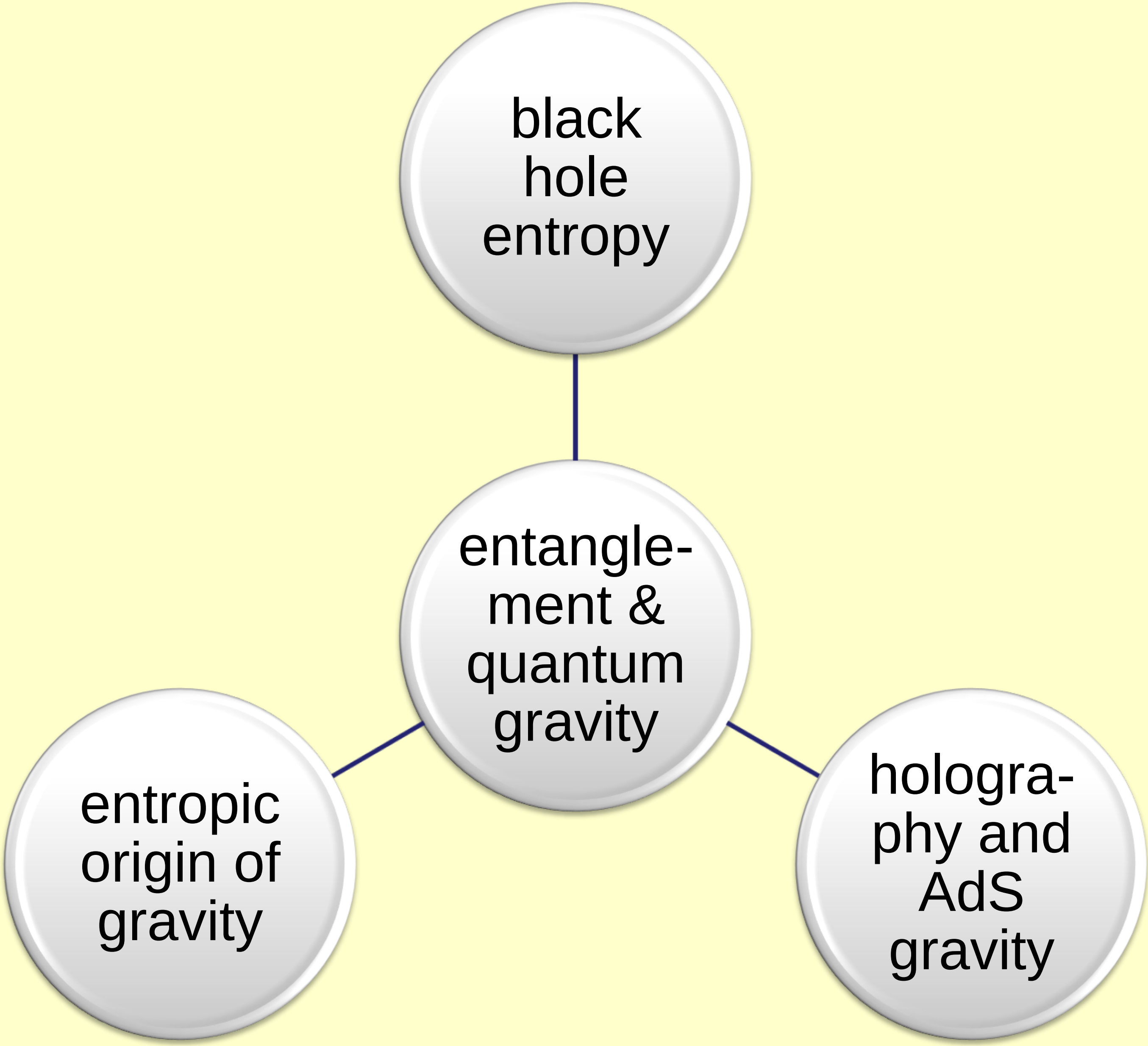

black hole entropy
entangle-ment & quantum gravity
entropic origin of gravity
hologra-phy and AdS gravity

Plan of lectures

Lecture 1: Entanglement entropy: definitions and computations

Lecture 2: Holographic entanglement entropy

Lecture 3: Entanglement entropy and emergent gravity

- Bekenstein-Hawking entropy of black holes in induced gravity
- Entanglement in Emergent gravity
- Entropy of wormholes
- Entanglement entropy in entropic gravity

A. Sakharov's suggestion (1968):

the Einstein theory can be induced at one-loop

$$\ln\det(\nabla_\mu \nabla^\mu + m^2) \simeq \frac{1}{16\pi G_{eff}} \int d^4x \sqrt{-g}\,(\Lambda_{eff} + R + a_{eff}\,"R^2" + \ldots)$$

$$\frac{\Lambda_{eff}}{G_{eff}} \sim M^4 \quad , \quad M - \text{UV cutoff}$$

$$\frac{1}{G_{eff}} \sim M^2$$

Gravitons = collective excitations of underlying degrees of freedom

analogy: phonons in solid state physics

Λ_{eff}, G_{eff} - Young's modulus

BH entropy in induced gravity (Frolov, Fursaev, Zelnikov):

$$L_S = -\nabla^2 + m_s^2 + \xi_s R \qquad N_s \text{ sclar fields}$$

$$L_d = i\gamma^\mu \nabla_\mu + m_d \qquad N_d \text{ spin 1/2 fields}$$

constraints: $q(0) = q(1) = 0$

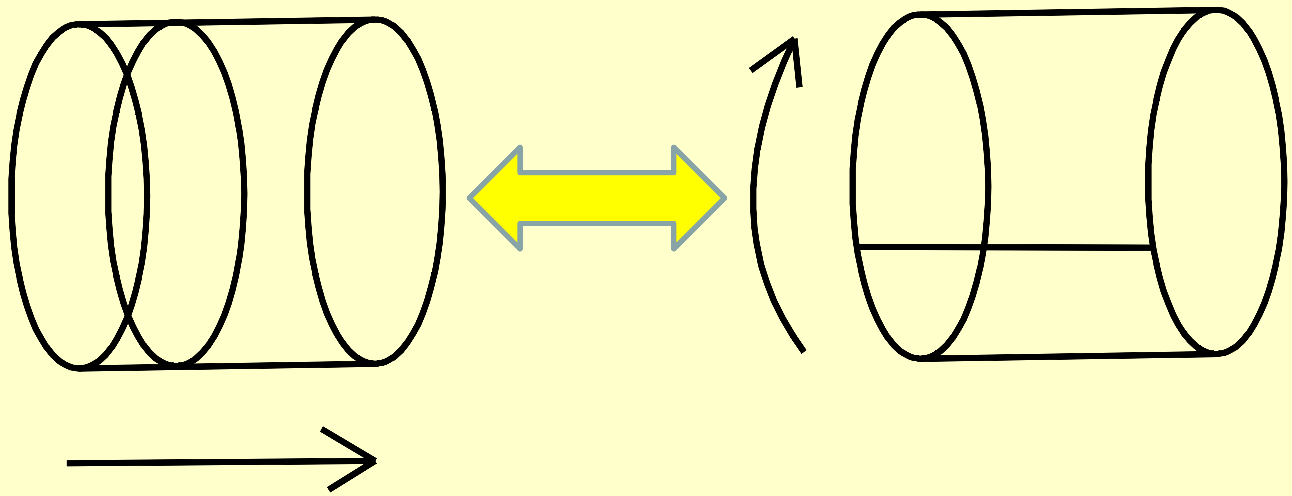
$$q(z) = \sum_k c_k m_k^{2z} \quad , c_d = 1, \quad c_s = 1 - 6\xi_s$$

$$\frac{1}{G_{eff}} = \frac{q'(1)}{12\pi} = \frac{1}{12\pi} \sum_k c_k m_k^2 \ln m_k^2$$

$$S^{BH} = S_{ent} - Q$$

Q is a Noether charge associated to non-minimal couplings

String theory:



"Tree-level" diagram
(closed strings)

"one-loop" diagram
(open strings)
"Sakharov's picture"

low-energy limit (10D (super)gravity, …)

Holographic formula enables one to compute entanglement entropy in strongly correlated systems with the help of classical methods (the Plateau problem)

What about entanglement in quantum gravity?

Can one define an entanglement entropy, S(B),
of fundamental degrees of freedom spatially separated by a surface B?

How can the fluctuations of the geometry be taken into account?

the hypothesis

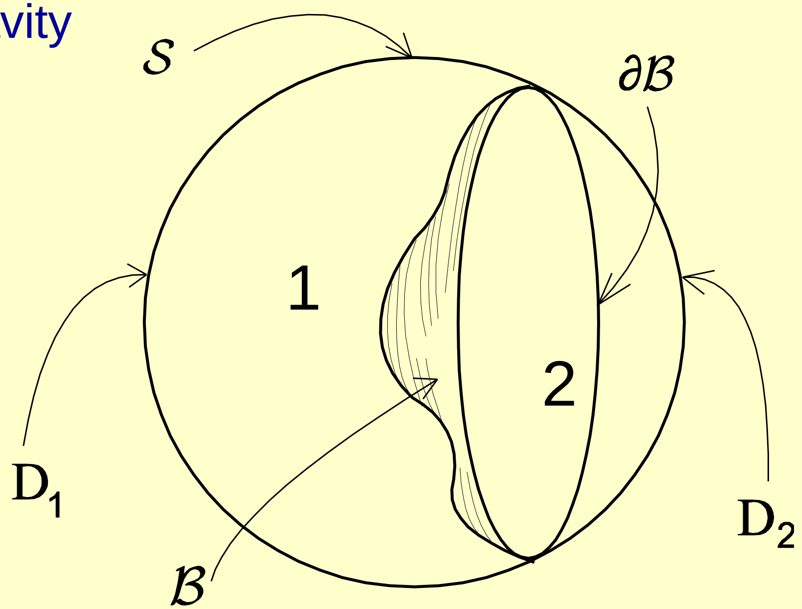
- S(B) is a macroscopical quantity (like thermodynamical entropy);

- S(B) can be computed without knowledge of a microscopical content of the theory (for an ordinary quantum system it can't)

- the definition of the entropy is possible at least for a certain type of boundary conditions

Suggestion (DF, 06,07): EE in quantum gravity between degrees of freedom separated by a surface B is

$$\boxed{S(B) = \frac{A(B)}{4G}}$$



B is a least area minimal hypersurface in a constant-time slice

conditions:

- static space-times

the system is determined by a set of boundary conditions; subsets, “1” and “2” , in the bulk are specified by the division of the boundary

The shape of the separating surface is
formed under fluctuations of the geometry;

As a result the surface is minimal, i.e. has a least area

Details: D.V. Fursaev, Phys. Rev. D77 (2008) 124002,
e-Print: arXiv:0711.1221 [hep-th]

● 'Proof' of this statement is equivalent to argument for Ryu-Takayanagi formula (see lecture 2);

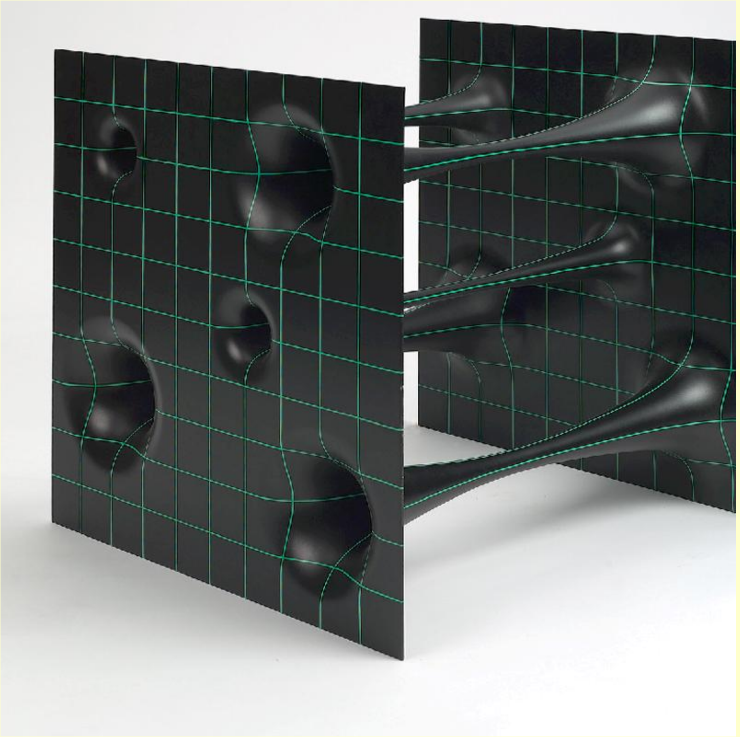
but quantum gravity 'lives' in 4D and allows a holographic description

● it is important that this suggestion satisfies the strong subadditivity condition (the proof is as in case of RT formula)

Some consequences

If the entanglement entropy in QG is a macroscopic quantity, does it allows a thermodynamic interpretation?

Wormhole is a shortcut inside a universe or a tunnel between two universes



$p_{II} = w_{II}\rho$ equation of state

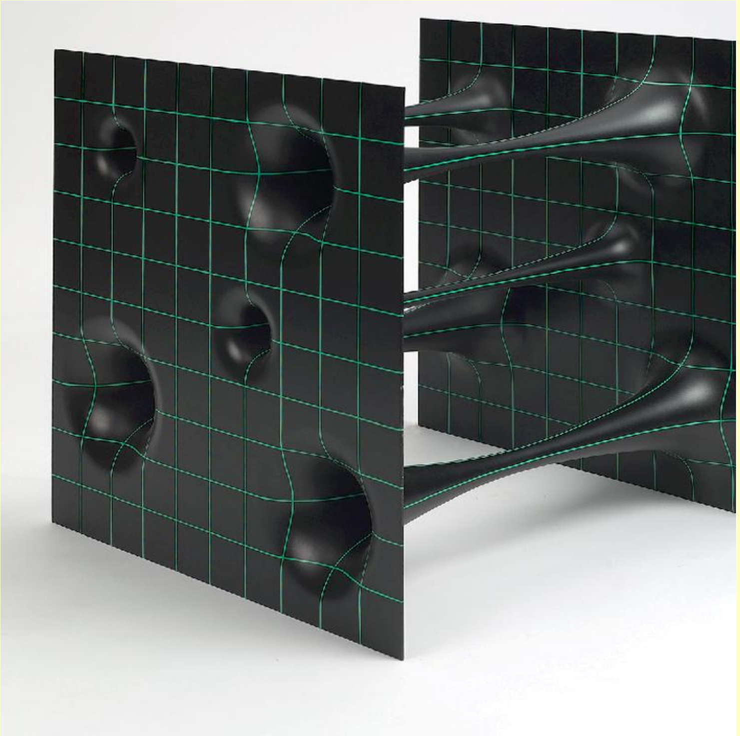
$w_{II} < -1$

p_{II} - pressure along the tunnel of WH

the null energy condition is violated

1st example of transversable WH is described by Morris and Thorne

Entropy of a wormhole



$$S_{\rm BH}(B) = S_{\rm ent}(B)$$ BH entropy = entanglement en.

B -minimal surface on the Einstein-Rosen bridge

$$S_{\rm WH}(B) = \frac{A(B)}{4G}$$ entropy of a wormhole

(see also Hayward, Martin-Moruno, Gonzalez-Diaz)

B - minimal surface in WH throat

Entanglement of states between two universes is determined by the area of a minimal surface at WH throat (“WH mouth”)

variational formula for a wormhole

$$ds^2 = -e^{2\varphi}dt^2 + \frac{dr^2}{1 - \frac{2E(r)}{r}} + r^2 d\Omega^2 = -2e^{2\varphi}dx^+ dx^- + r^2 d\Omega^2$$

$r = r_H : \quad E(r) = \dfrac{r}{2}$ - position of the WH mouth

$$\partial_r E = 4\pi r^2 \rho$$

$\delta E = \dfrac{\kappa}{2\pi}\delta S + w\delta V$ - an identity where:

$E \equiv E(r_H) = \dfrac{1}{2} r_H;$ - a "mass" of a WH

$$S = \frac{A(r_H)}{4G}; \quad V \equiv \frac{4\pi r_H{}^3}{3}; \quad \kappa \equiv \frac{1}{2r_H} - 4\pi r_H w$$

w - is a parameter

one can put:

$$w = e^{-2\varphi} T_{+-} = -\frac{1}{2} Tr_{(2)} T_{\mu\nu}$$

$T_{\mu\nu}$ - stress-energy tensor of the matter on the mouth

$w\delta V$ a “work term”

$\kappa = -e^{-2\varphi} \partial_+ \partial_- r$ - a “surface gravity”

$$\boxed{\delta E = \frac{\kappa}{2\pi} \delta S + w\delta V}$$

-an analog of the 1st law
(in the Hayward form)

These formulae can be extended to non-static cases, WH mouth being a temporal trapping horizon (a marginal sphere)

Dynamics of non-static spherically symmetric wormholes:

S. Hayward 0903.5438 [gr-qc];

P. Martin-Moruno and P. Gonzalez-Diaz 0904.0099 [gr-qc]

application to a charged black hole

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2$$

$$B(r) = 1 - \frac{2MG}{r} + \frac{q^2}{r^2}$$

$$\kappa = -e^{-2\varphi}\partial_+\partial_- r = \frac{1}{2}\partial_r B(r_H)$$

- the surface gravity

$$w = \frac{1}{2}F_{0r}^2 = \frac{q^2}{8\pi r_H{}^4}$$

$$\delta E = \frac{\kappa}{2\pi}\delta S + w\delta V$$

-1st law for a charged black hole
(a non-standard form)

$$E = M - \frac{q^2}{2r_H}$$

- Misner-Sharp energy

reasonings based on the universality of the variational formulae (the “1st law”) for black holes and wormholes suggest that

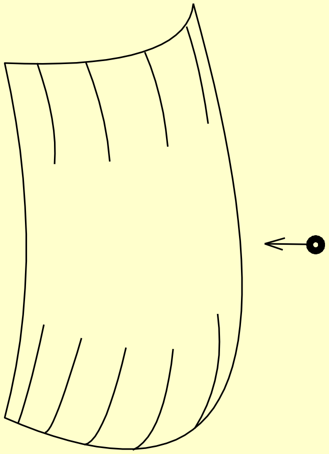
wormholes are characterized by an entropy

WH entropy measures entanglement of quantum states
in the univereses connected by the WH tunnel

analogous conclusion based on variational formulae:
S. Hayward,

P. Martin-Moruno and P. Gonzalez-Diaz

simple variational formulae
(weak field approximation)



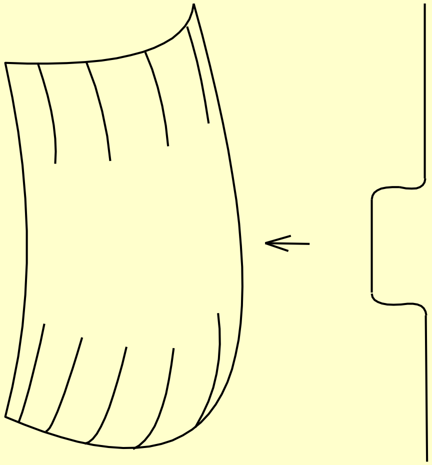
$$\delta S = \pi m l$$

m – mass of a particle

l – shift (toward of the surface)

$$\delta S \sim 10^{37} \quad \text{if} \quad m = 1g, \quad l = 1cm$$

$\delta S = O(1)$ if l is a Compton wavelength



$$\delta S \simeq \pi \mu l \delta z$$

μ – string tension

δz – lenght of the segment

Entropic origin of gravity (E. Verlinde)

Consider a massive source and a holographic screen around it;

1[st] postulate the screen is equipotential surface which carries certain entropy:

$$dN = \frac{d\sigma}{G} -$$ number of degrees of freedom on the screen on the area $d\sigma$

2d postulate: $\boxed{\delta S = 2\pi m l}$ - change of the entropy under the movement of a test particle toward the screen;

3d postulate: the energy takes an 'equipartition' form on the screen

$$\boxed{M = \frac{1}{4\pi G}\int d\sigma \partial_n \phi = \frac{1}{2}\int T dN}$$

(T. Padmanabhan)

$$T = \frac{\partial_n \phi}{2\pi} -$$ a local temperature on the screen

E.Verlinde
arXiv:1001.0785 [hep-th]

Consequences

use an analog of the 1st law $\boxed{T\delta S = W}$

$W = Fl -$ a work done by the system,

$F = mw$ - force acting on the test particle;

w - acceleration of the particle

$$\boxed{F = mw = m\partial\phi = \frac{mMG}{r^2} \text{ - the Newton law}}$$

gravity is an emergent phenomenon;
the force of gravity has an entropic origin
direction of the force – gradients of the entropy

aim for the rest of 3d lecture

to study simplest dynamics of a minimal surface;

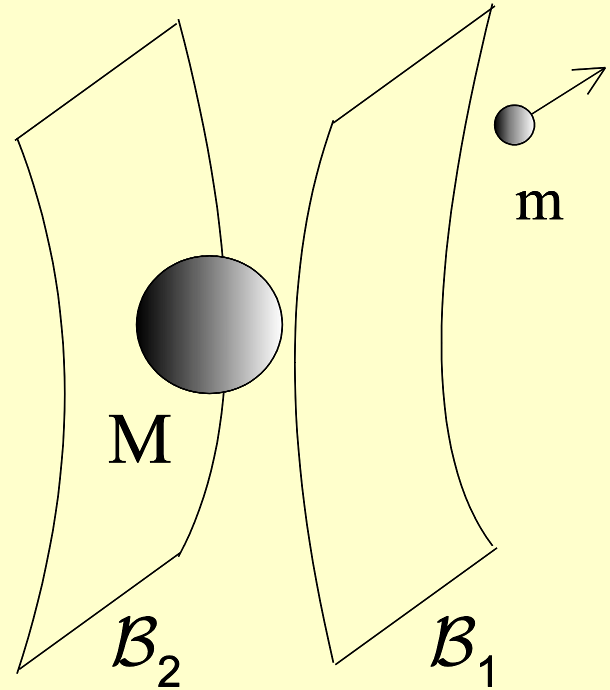
to look for its thermodynamic analogy;

to relate this analysis to a hypothesis about an entropic origin of gravity (as suggested by E.Verlinde)

see D.V. Fursaev, **arXiv:1006.2623** [hep-th]

Minimal surfaces may play
a role of holographic screens

2-component 'screen' around
a massive source



in weak field approximation
screen = 2 parallel planes

$$ds^2 = -(1 + 2\phi)\, dt^2 + (1 - 2\phi)\,(dx^2 + dy^2 + dz^2), \quad \phi = -\frac{2MG}{r}$$

$$E(B_k) = \frac{1}{4\pi MG}\int d\sigma w_n\,, \quad w_n = \partial_n \phi - \text{ acceleration on the screen}$$

$$E = E(B_1) + E(B_2) = M - \text{ the Komar energy of the source}$$

Dynamics in the weak field approximation

$$S_k = \frac{A(B_k)}{4G} = \overline{S_k} - \frac{1}{2}\int \phi(r_k)\,dN;$$ $\overline{S_k}$ – entropy for a plane

$\phi(r) = -\frac{MG}{r}$ - potential of the massive source

$A'(B_k) = -2\int \phi'(r'_k)\,d\sigma$ - modification of the area by a test particle

$\phi'(r'_k) = -\frac{mG}{r'_k}$ - potential of the test particle on the screen

r'_k - distance from a point on the screen to the particle

$$\boxed{\delta A(B_k) = -2\int \delta\phi'(r'_k)\,d\sigma = -4\pi m G l}$$

Notes on the computation

$$\phi'(r) = 4\pi mG\, D(x_r - x_0)$$

x_0 - position of the particle, x_r - position of a point on a screen

$$\Delta D(x) = \delta(x_0)$$

shift of the particle $(\delta x_0)^k = l^k$ results in the variation

$$\delta_0 D(x_r - x_0) = -l^k \partial_k D(x_r - x_0)$$

$$\int \delta\phi'(r')\, d\sigma \quad \to$$

$$\int l^k \partial_k D(x_r - x_0) d\sigma = l_\perp \int \Delta D(x) d^3 x = \frac{l}{2}$$

$l \equiv l_\perp$ shift in the direction orthogonal to the screen

shifts along the screen (plane) do not change the area

`Thermodynamics’

$\delta S_k = \delta S(B_k) = -\pi m l$ - for a particle moving out of the surface

$\delta S = 0$ - for a particle moving inside the screen

$\boxed{\delta S = \delta S_1 + \delta S_2 = -2\pi m l}$ - can be derived, is not a postulate!

single surface = half of the screen:

$$E(B) = \frac{1}{4\pi M G}\int_B d\sigma w_n = \frac{M}{2}$$

energy balance:

$$\boxed{T(x)\delta S(B) = -\frac{1}{2}\delta W(x) \;\rightarrow T(x) = \frac{w_n}{2\pi} \;\rightarrow E(B) = \frac{1}{2}\int_B T dN}$$

$\delta W(x)$ – work done by an external force to drag the test particle with coordinates x out of the surface

Static space-time backgrounds
(which are solutions to the Einstein equations)

$$ds^2 = g_{00}(x)dt^2 + g_{ab}(x)dx^a dx^b$$

`holographic screen` is a minimal surface (with a topology of a hyperplane) in a constant-time slice

$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ - perturbation caused by a test particle

$A(B) \to A(B) + A'(B)$ - perturbation of the area of a minimal surface

$$\boxed{A'(B) = \frac{1}{2}\int_B d\sigma\, \gamma^{ij} X^a{}_{,i} X^b{}_{,j} h_{ab}}$$

$X^a = X^a(y)$- change in position of the surface does not count in the linear approximation

$\gamma^{ij} X^a{}_{,i} X^b{}_{,j} h_{ab}$ - variation of the metric induced on the surface

perturbations:

$$L\bar{h}_\nu^\mu = 16\pi G\, t^\mu{}_\nu \quad , \quad \bar{h}_\nu^\mu = h^\mu{}_\nu - \frac{1}{2} h \delta^\mu{}_\nu$$

$$\nabla_\mu \bar{h}_\nu^\mu = 0$$

$$L\bar{h}_\nu^\mu = -\nabla^2 \bar{h}_\nu^\mu - 2R_{\nu\rho}{}^{\mu\lambda} \bar{h}_\lambda^\rho - \frac{8\Lambda}{n-2} \bar{h}_\nu^\mu$$

Λ - cosmological constant, $n -$ number of dimensions

$$\nabla^2 \bar{h}_0^0 = \Delta \bar{h}_0^0 - w^a \partial_a \bar{h}_0^0 + 2w^2 \bar{h}_0^0 - 2w^a w_b \bar{h}_a^b$$

Δ - Laplacian on constant-time slice

$t^\mu{}_\nu(x) = m u^\mu u_\nu \delta^{(n-1)}(x, x_0)$, m - mass of the particle

u^μ - velocity of the particle , $w^\mu = \nabla u^\mu$ - acceleration

approximation:

curvature terms, lambda term, and acceleration terms are “slowly” changing,

perturbations caused by the particle are rapidly changing ;

curvature-, lambda-, and acceleration terms can be neglected

$$A'(B) = \frac{1}{2}\int_B d\sigma\, \bar{h}_0^{\,0} - \text{area perturbation}$$

$$\bar{h}_0^{\,0} = 16\pi G m\, D(x, x_0)$$

$$\Delta_x D(x, x_0) = \delta^{(n-1)}(x, x_0)$$

an approximate translation invariance in z-direction:

$$D(z,y;z',y') \simeq D(z-z',y;y')$$

$$dl^2 = dz^2 + \gamma_{ij}(z,y)dy^i dy^j$$

$z = 0 -$ position of the surface

$k = 0 -$ extrinsic curvature of the minimal surface $\rightarrow$

$$\partial_z \det\gamma_{ij}(0,y) = 0 \rightarrow \Delta = \partial_z^2 + \tilde{\Delta}(z,y) \simeq \partial_z^2 + \tilde{\Delta}(0,y)$$

$\delta A(B) = \delta A'(B) = -4\pi m G l -$ for the shift of the particle out of the surface by distance l

a universal formula, does not depend on the background and its dimensionality

`Thermodynamical' parameters of a minimal surfaces

$$S=\frac{A(B)}{4G}-\text{entropy}$$

$$\boxed{E=\frac{1}{4\pi G}\int_B |g_{00}|^{1/2} w_n d\sigma=\frac{1}{2}\int_B TdN}-\text{energy (Komar mass)}$$

$$dN=\frac{d\sigma}{G}-\text{number of states on the area}$$

$$w_n-\text{acceleration at the surface (part normal to the surface)}$$

$$\boxed{T=\frac{|g_{00}|^{1/2} w_n}{2\pi}}--\text{local temperature on the surface}$$

temperature coincides with the Hawking temperature for the surface located near a back hole horizon

`Thermodynamics'

the surface is located between a gravitating body
and a test particle

one obtains "1st law" $T(x)\delta S = -\frac{1}{2}\delta W(x)$

$\delta S = -\pi ml -$ entropy change when dragging particle out of the surface

$\delta W(x) = Fl = m|g_{00}|^{1/2} w_n$

$F -$ force applied by an observer at infinity

(for asymptotically flat spacetimes)

thank you for attention