Entanglement thermodynamics

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Plan of the talk

- Review of (holographic) entanglement entropy
- Thermalization and entanglement entropy
- Laws of entanglement thermodynamics
- Summary

References

T. Nishioka, S. Ryu and T. Takayanagi, "Holographic Entanglement Entropy: An Overview," arXiv: 0905.0932.

First part

H. Liu and S. J. Suh, "Entanglement growth during thermalization in holographic systems," arXiv:1311.1200 [hep-th].

M. A., A. F. Astaneh and M. R. M. Mozaffar, "Thermalization in Backgrounds with Hyperscaling Violating Factor," arXiv:1401.2807 [hep-th].

P. Fonda, L. Franti, V. Keranen, E. Keski-Vakkuri, L. Thorlacius and E. Tonni, "Holographic thermalization with Lifshitz scaling and hyperscaling violation," arXiv:1401.6088 [hep-th].

Second Part

J. Bhattacharya, M. Nozaki, T. Takayanagi and T. Ugajin, "Thermodynamical Property of Entanglement Entropy for Excited States," arXiv:1212.1164.

M.A. , D. Allahbakhshi and A. Naseh, "Entanglement Thermodynamics," arXiv:1305.2728 .

D. D. Blanco, H. Casini, L. -Y. Hung and R. C. Myers, "Relative Entropy and Holography," arXiv:1305.3182.

T. Faulkner, M. Guica, T. Hartman, R. C. Myers and M. Van Raamsdonk, "Gravitation from Entanglement in Holographic CFT's," arXiv:1312.7856.

Entanglement entropy

Consider a state $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, which evolves in time by its Hamiltonian H

Physical quantities are computed as expectation values of operators as follows

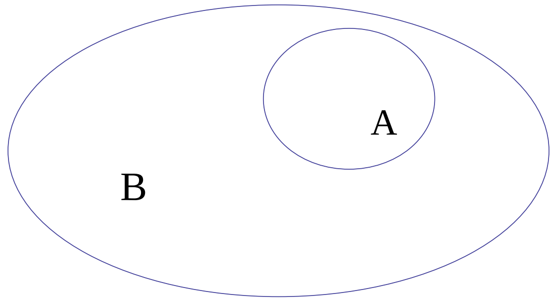
$$\langle O\rangle = \langle\psi|O|\psi\rangle = \mathrm{Tr}(\rho O)$$

where we defined the density matrix $\rho = |\psi\rangle\langle\psi|$. This system is called a pure state as it is described by a unique wave function $|\psi\rangle$.

In mixed states, the system is described by a density matrix ρ. An example of a mixed state is the canonical distribution

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$$

Assume that the quantum system has multiple degrees of freedom and so one can decompose the total system into two subsystems A and B



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

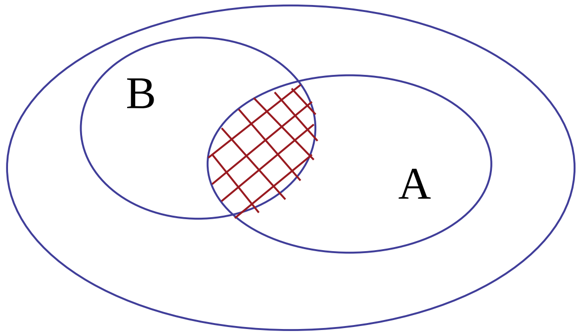
The reduced density matrix of the subsystem A

$$\rho_A = \mathrm{Tr}_B(\rho)$$

Then the entanglement entropy is defined as the von-Neumann entropy for A

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

Properties of Entanglement entropy



1. For pure state

$$S_A = A_B$$

2. For two subspace A and B, the strong subadditivity is

$$S_A + S_B \leq S_{A\cup B} + S_{A\cap B}$$

3. Leading divergence term is proportional to the area of the boundary ∂A

$$S_A = c_0 \frac{\text{Area}}{\epsilon^{d-1}} + O(\epsilon^{-(d-2)}),$$

where c_0 is a numerical constant; ϵ is the ultra-violet(UV) cut off in quantum field theories.

Rényi entropies

It is also useful to compute Rényi entropies

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}\rho^n$$

Then the entanglement entropy is given by

$$S_E = \lim_{n\to 1} S_n$$

Practically one may first compute $\mathrm{Tr}(\rho^n)$ by making use the replica trick and then

$$S_E = -\partial_n \mathrm{Tr}\rho^n|_{n=1}$$

AdS/CFT correspondence

Basically AdS/CFT correspondence is a duality or a relation between two theories one with a gravity and the other without gravity.

The gravitational theory is usually defined in higher dimension.

Well developed case is the one where the gravity is defined on an AdS geometry where the dual theory is a CFT living in the conformal boundary of AdS space.

Classical gravity on an asympotically locally AdS$_{d+1}$ background is dual to a d-dimensional Large N strongly coupled field theory with a UV fixed point on its boundary.

AdS$_{d+1}$ metric in Poincare coordinates

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2.$$

AdS$_{d+1}$ metric in global coordinates

$$ds^2 = -(1 + \frac{r^2}{R^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2 d\Omega_{d-1}^2.$$

Here boundary is at $r \to \infty$

There is one to one correspondence between objects in CFT and those in the gravitational theory on AdS space.

Gravity	$\Longleftrightarrow$	Field theory
$\{r, t, \vec{x}\}$	$\Longleftrightarrow$	$\{\text{scale of energy}, t, \vec{x}\}$
Near boundary Near horizon	$\Longleftrightarrow$	UV, IR regions
Symmetries	$\Longleftrightarrow$	Symmetries
Fields $\Phi(r, t, \vec{x})$	$\Longleftrightarrow$	Operators $\mathcal{O}(t, \vec{x})$
On shell action	$\Longleftrightarrow$	Generating function

Suppose $\mathcal{O}(t, \vec{x})$ on the boundary corresponds to field $\Phi(r, t, \vec{x})$ in the bulk whose action is $S[\Phi]$.

Suppose $\mathcal{O}(t, \vec{x})$ has dimension Δ:

$$\mathcal{O}(x) = \lambda^{\Delta} \mathcal{O}(x)$$

One can solve equations of motion of $\Phi(r, t, \vec{x})$ in the bulk.

The asymptotic expansion is

$$\lim_{r \to 0} \Phi(r, t, \vec{x}) \sim r^{d-\Delta} \varphi(t, \vec{x}) + r^{\Delta} \phi(t, \vec{x}).$$

Δ is given in terms of mass, dimension,....

$\varphi(t, \vec{x})$ is source and $\phi(t, \vec{x})$ is response.

$\varphi(t, \vec{x})$ is source

$$\int d^d x \mathcal{L}_{FT} + \int d^d x \; \varphi(t, \vec{x}) \mathcal{O}(t, \vec{x})$$

$\phi(t, \vec{x})$ is response

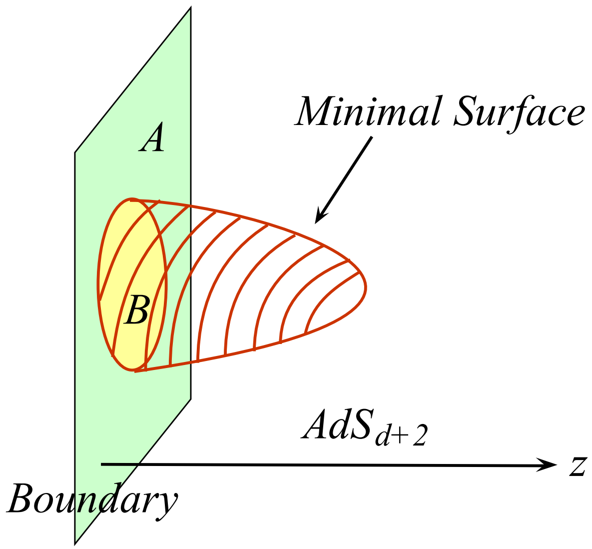
$$\langle \mathcal{O} \rangle = \frac{1}{\sqrt{g}} \frac{\delta S[\varphi]}{\delta \varphi} \sim \phi[\varphi] + \text{local terms.}$$

n-point function

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \sim \left. \frac{\delta^{n-1} \phi[\varphi]}{\delta \varphi^{n-1}} \right|_{\varphi=0}$$

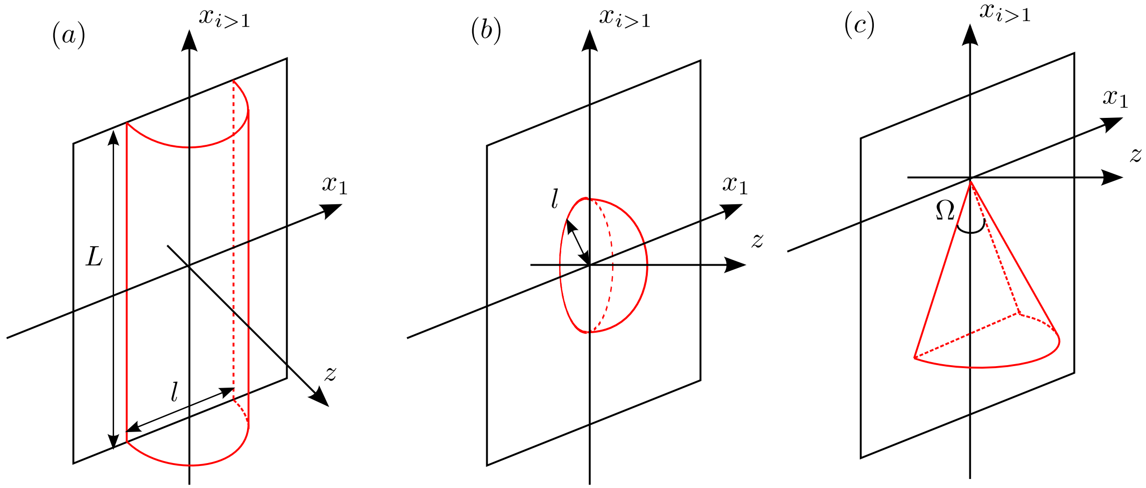
Holographic Formula for Entanglement Entropy

For static background and fixed time divide the boundary into A and B. Extend this division $A \cup B$ to of the bulk spacetime. Extend ∂A to a surface γ_A in the entire spacetime such that $\partial\gamma_A = \partial A$.



$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}$$

S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT," Phys. Rev. Lett. **96**, 181602 (2006) [hep-th/0603001].



Consider a strip in a d dimensional CFT at fixed time

$$t = \text{fixed}, \qquad -\frac{\ell}{2} \leq x_1 \leq \frac{\ell}{2}, \qquad 0 \leq x_i \leq L, \quad i = 2, \cdots, d-1.$$

$$dS^2 = \frac{R^2}{z^2}(-dt^2 + dz^2 + dx_1^2 + dx_i^2), \qquad z = \frac{1}{r}$$

Consider a profile in the bulk $x_1 = x(z)$, so that the induced metric reads

$$dS^2_{\text{ind}} = \frac{R^2}{z^2}[(1 + x'^2)dz^2 + dx_i^2].$$

The area of the induced metric is

$$A = L^{d-2} R^{d-1} \int dz \, \frac{\sqrt{1 + x'^2}}{z^{d-1}}$$

One needs to minimize the area

$$\frac{x'}{z^{d-1}\sqrt{1 + x'^2}} = \text{constant} = \frac{1}{z_t^{d-1}}$$

The width and the entanglement entropy are

$$\frac{\ell}{2} = \int_0^{z_t} dz \, \frac{(z/z_t)^{d-1}}{\sqrt{1 - (\frac{z}{z_t})^{2(d-1)}}}, \qquad S = \frac{L^{d-2} R^{d-1}}{2G} \int_\epsilon^{z_t} \frac{dz}{z^{d-1}\sqrt{1 - (\frac{z}{z_t})^{2(d-1)}}},$$

where z_t is a turning point and ϵ is a UV cut-off.

$$S = \begin{cases} \frac{L^{d-2}R^{d-1}}{2G}\left(-\frac{1}{(d-1)\epsilon^{d-2}} + \frac{c_0}{\ell^{d-2}}\right) & \text{for } d \neq 2, \\ \\ \frac{R}{2G}\ln\frac{\ell}{\epsilon}, & \text{for } d = 2, \end{cases}$$

with c_0 being a numerical factor

$$c_0 = \frac{2^{d-2}\pi^{\frac{d-1}{2}}}{d-2}\left(\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}\right)^{d-1}$$

For holographic entanglement entropy

1. The formula leads to the area law (for Einstein gravity).

2. The strong subadditivity can also be holographically proven (for static background)

3. For 2D CFT using AdS_3 one finds

$$S_A = \frac{c}{3} \ln \frac{\ell}{\epsilon}$$

where ℓ width of strip, $c = \frac{3R}{2G}$.

Higher Derivative Corrections

The holographic formula we have consudered is for Einstein gravity. Motivated by the Wald formula it is interesting to see how this formula is modified in the presence of higher derivative corrections to Einstein gravity.

Unlike the Wald formula there no a general expression when we have arbitrary higher derivative corrections. However the holographic entanglement entropy has been found only for the Lovelock gravities

For the Gauss-Bonnet gravity whose gravity action is

$$S_{GB} = -\frac{1}{16G_N}\int d^{d+2}x\sqrt{g}\left[R - 2\Lambda + \lambda(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\right],$$

The holographic entanglement entropy is argued to be

$$S_A = \mathrm{Min}_{\gamma_A}\left[\frac{1}{4G_N}\int_{\gamma_A} d^dx\sqrt{h}(1+2\lambda R_{int})\right],$$

where R_{int} is the intrinsic curvature of γ_A.

L. Y. Hung, R. C. Myers and M. Smolkin, On Holographic Entanglement Entropy and Higher Curvature Gravity, JHEP 1104 (2011) 025 [arXiv:1101.5813 [hep-th]].

J. de Boer, M. Kulaxizi and A. Parnachev, Holographic Entanglement Entropy in Lovelock Gravities, JHEP 1107 (2011) 109 [arXiv:1101.5781 [hep-th]].

For higher derivative see for examples

D. V. Fursaev, A. Patrushev and S. N. Solodukhin, “Distributional Geometry of Squashed Cones,” arXiv:1306.4000 [hep-th].

A. Bhattacharyya, M. Sharma and A. Sinha, “On generalized gravitational entropy, squashed cones and holography,” arXiv:1308.5748 [hep-th].

X. Dong, “Holographic Entanglement Entropy for General Higher Derivative Gravity,” arXiv:1310.5713 [hep-th].

J. Camps, “Generalized entropy and higher derivative Gravity,” arXiv:1310.6659 [hep-th].

See also Faraji and Mohammado Mozaffar’ talks

Part one

Time-dependent backgrounds

So far we have considered static case where we have a time slice on which we can define minimal surfaces. In the time-dependent case there is no a natural choice of the time-slices.

In Lorentzian geometry there is no minimal area surface. In order to resolve this issue we use the covariant holographic entanglement entropy which is

$$S_A(t) = \frac{\text{Area}(\gamma_A(t))}{4G_N^{(d+2)}}$$

where $\gamma_A(t)$ is the extremal surface in the bulk Lorentzian spacetime with the boundary condition $\partial\gamma_A(t) = \partial A(t)$.

Strong subadditivity?

V. E. Hubeny, M. Rangamani and T. Takayanagi, "A Covariant holographic entanglement entropy proposal," JHEP **0707**, 062 (2007) [arXiv:0705.0016 [hep-th]].

Example of time-dependent case: Black hole formation or Thermalization

$$\text{Geometry} \Longleftrightarrow \text{State}$$

$$AdS \text{ solution} \Longleftrightarrow \text{Vaccum state}$$
$$\text{Black hole} \Longleftrightarrow \text{Excited state; thermal}$$

Let us perturbe a system so that the end point of the time evolution would be a thermal state. This might be done by a global quantum quench. Typically during evolution the system is out of equilibrium.

The thermalization process after a global quantum quench may be map to a black hole formation due to a gravitational collapse.

A quantum quench in the field theory may occurs due to a sudden change in the system which might be caused by turning on the source of an operator in an interval $\delta t \to 0$.

This change can excite the system to an excited state with non-zero energy density that could eventually thermalize to an equilibrium state.

From gravity point of view this might be described by a gravitational collapse of a thin shell of matter which can be modelled by an AdS-Vaidya metric.

$$dS^2 = \frac{R^2}{r^2}[f(r,v)dv^2 - 2drdv + d\vec{x}^2], \quad f(\rho, v) = 1 - m\theta(v)r^d$$

where r ρ is the radial coordinate, x_is are spatial boundary coordinates and v is the null coordinate. Here $\theta(v)$ is the step function and therefore for $v < 0$ the geometry is an AdS metric while for $v > 0$ it is an AdS-Schwarzschild black hole.

$$S=-\frac{1}{16\pi G_N}\int d^{D+2}x\sqrt{-g}\left[R-\frac{1}{2}(\partial\phi)^2+V(\phi)-\frac{1}{4}\sum_{i=1}^{N_g}e^{\lambda_i\phi}F^{(i)2}\right]+S_{\text{matter}},$$

where $V(\phi)=V_0e^{\gamma\phi}$, G is the Newton constant, γ, V_0 and λ_i are free parameters of the model.

$$ds^2=r^{-2\frac{\theta}{D}}\left(-r^{2z}f(r,v)dv^2+2r^{z-1}drdv+r^2d\vec{x}^2\right),\quad \phi=\beta\ln r,$$

$$A_v^{(1)}=\sqrt{\frac{2(z-1)}{D-\theta+z}}\,r^{D-\theta+z},\qquad A_v^{(2)}=\sqrt{\frac{2(D-\theta)}{D-\theta+z-2}}\,\frac{Q(v)}{r^{D-\theta+z-2}}.$$

where z is the dynamical exponent and θ is the hyperscaling violation exponent.

$$f(r,v)=1-\frac{m(v)}{r^{D-\theta+z}}+\frac{Q(v)^2}{r^{2(D-\theta+z-1)}},$$

The energy momentum and current density of the charged infalling matter are given by $T_{\mu\nu} = \varrho U_\mu U_\nu$ and $J_\mu^{(2)} = \varrho_e U_\mu$ with $U_\mu = \delta_{\mu v}$, and

$$\varrho = \frac{\theta - D}{2} \frac{\partial f(r,v)}{\partial v}\, r^z, \qquad \varrho_e = \frac{\partial Q(v)}{\partial v} \sqrt{2(D-\theta)(D-\theta+z-2)}\, r^{\theta - D}.$$

Note that the null energy condition requires $\varrho > 0$.

In what follows $d = D - \theta + 1$ is the effective dimension.

To compute the entanglement entropy for a strip with width ℓ, let us consider the following strip

$$-\frac{\ell}{2} \leq x_1 = x \leq \frac{\ell}{2}, \qquad 0 \leq x_a \leq L, \quad \text{for } a = 2, \cdots, D.$$

Since the metric is not static one needs to use the covariant proposal for the holographic entanglement entropy. Therefore the corresponding co-dimension two hypersurface in the bulk may be parametrized by $v(x)$ and $\rho(x)$. Then the induced metric on the hypersurface is

$$ds^2_{\text{ind}} = \rho^{2\frac{1-d}{D}}\left[\left(1 - \rho^{2-2z} f(\rho, v) v'^2 - 2\rho^{1-z} v' \rho'\right) dx^2 + dx_a^2\right),$$

The area of the hypersurface reads

$$\mathcal{A} = \frac{L^{D-1}}{2} \int_{-\ell/2}^{\ell/2} dx \, \frac{\sqrt{1 - 2\rho^{1-z} v' \rho' - \rho^{2-2z} v'^2 f}}{\rho^{d-1}}$$

We note, however, that since the action is independent of x the corresponding Hamiltonian is a constant of motion

$$\rho^n \mathcal{L} = H = \text{constant.}$$

Moreover we have two equations of motion for v and ρ. Indeed, by making use of the above conservation law the corresponding equations of motion read

$$\partial_x P_v = \frac{P_\rho^2}{2} \frac{\partial f}{\partial v}, \qquad \partial_x P_\rho = \frac{P_\rho^2}{2} \frac{\partial f}{\partial \rho} + \frac{n}{\rho^{2n+1}} H^2 + \frac{1-z}{\rho^{2-z}} P_\rho P_v,$$

where

$$P_v = \rho^{1-z}(\rho' + \rho^{1-z} v' f), \qquad P_\rho = \rho^{1-z} v',$$

are the momenta conjugate to v and ρ up to a factor of H^{-1}, respectively.

These equations have to be supplemented by the following boundary conditions

$$\rho(\frac{\ell}{2}) = 0, \qquad v(\frac{\ell}{2}) = t, \qquad \rho'(0) = 0, \qquad v'(0) = 0,$$

and

$$\rho(0) = \rho_t, \qquad v(0) = v_t,$$

where (ρ_t, v_t) is the coordinate of the extremal hypersurface turning point in the bulk.

In what follows we will consider the case of $\ell \gg \rho_H$

The process we will be considering for the thermalization after a global quantum quench consists of three phases: initial phase, intermediate phase and final phase.

i) $v<0$ region

In this case which corresponds to the vacuum solution one has

$$P_{(i)v}=\rho'+\rho^{1-z}v'=0,$$

which together with the conservation law yields to

$$v(\rho)=v_t+\frac{1}{z}(\rho_t^z-\rho^z),\qquad x(\rho)=\int_\rho^{\rho_t}\frac{d\xi\;\xi^n}{\sqrt{\rho_t^{2n}-\xi^{2n}}}.$$

Note also that at the null shell where $v=0$, from the above equation, one gets

$$\rho_c^z=\rho_t^z+zv_t$$

which, indeed, gives the point where the extremal hypersurface intersects the null shell. Moreover, from the conservation law in the initial phase one finds

$$\rho'_{(i)}=-\rho_c^{1-z}v'_{(i)}=-\sqrt{\left(\frac{\rho_t}{\rho_c}\right)^{2n}-1}$$

ii) $v > 0$ region

In this case which the corresponding geometry is a the black hole, using the conservation law one arrives at

$$\rho'^2 = \frac{P^2_{(f)v}}{\rho^{2-2z}} + \left(\left(\frac{\rho_t}{\rho}\right)^{2n} - 1\right) \tilde{f}(\rho) \equiv V_{eff}(\rho),$$

which can also be used to find

$$\frac{dv}{d\rho} = -\frac{1}{\rho^{2(1-z)}\tilde{f}(\rho)} \left(\rho^{1-z} + \frac{P_{(f)v}}{\sqrt{V_{eff}(\rho)}}\right).$$

Here $V_{eff}(\rho)$ might be thought of as an effective potential for a one dimensional dynamical system whose dynamical variable is ρ. In particular the turning point of the potential can be found by setting $V'_{eff}(\rho) = 0$.

iii) Matching at the null shell

blue Since ρ and v are the coordinates of the space time they should be continuous across the null shell.

We note. however, that since one is injecting matters along the null direction v, one would expect that its corresponding momentum conjugate jumps once one moves from the initial phase to the final phase.

Therefore by integrating the equations of motion across the null shell one arrives at

$$\rho'_{(f)} = \left(1 - \frac{1}{2} g(\rho_c)\right) \rho'_{(i)}, \quad \mathcal{L}_{(f)} = \mathcal{L}_{(i)}, \quad v'_{(f)} = v'_{(i)}.$$

It is, then, easy to read the momentum conjugate of v in the final phase

$$P_{(f)v} = \frac{1}{2} \rho_c^{1-z} g(\rho_c) \rho'_{(i)} = -\frac{1}{2} \rho_c^{1-z} g(\rho_c) \sqrt{\left(\frac{\rho_t}{\rho_c}\right)^{2n} - 1}.$$

Now we have all ingredients to find the area of the corresponding extremal hypersurface in the bulk. In general the extremal hypersurface could extend in both $v < 0$ and $v > 0$ regions of space-time. Therefore the width ℓ and the boundary time are found

$$\frac{\ell}{2} = \rho_t \left(\int_{\frac{\rho_c}{\rho_t}}^{1} \frac{d\xi \; \xi^{d-1}}{\sqrt{1 - \xi^{2(d-1)}}} + \int_0^{\frac{\rho_c}{\rho_t}} \frac{d\xi}{\sqrt{R(\xi)}} \right), \qquad t = \rho_t^z \int_0^{\frac{\rho_c}{\rho_t}} \frac{d\xi \; \xi^{z-1}}{h(\xi)} \left(1 + \frac{\xi^{z-1} E}{\sqrt{R(\xi)}} \right),$$

where $E = P_{(f)v} \rho_t^{z-1}$.

Finally the entanglement reads

$$S = \frac{L^{d-2}}{2G\rho_t^{d-2}} \left(\int_{\frac{\rho_c}{\rho_t}}^{1} \frac{d\xi}{\xi^{d-1}\sqrt{1 - \xi^{2(d-1)}}} + \int_0^{\frac{\rho_c}{\rho_t}} \frac{d\xi}{\xi^{2(d-1)}\sqrt{R(\xi)}} \right).$$

- Early times growth where $t \ll \rho_H^z$

$$\Delta S \approx \frac{L^{D-1} m}{4G(z+1)} (zt)^{1+\frac{1}{z}},$$

- The intermediate region where $\frac{\ell}{2} \gg t \gg \rho_H^z$

$$\Delta S = L^{D-1} S_{\text{th}}\, v_E\, \rho_H^{1-z} t,$$

where

$$v_E = \left(\frac{d+z-3}{2(d+z-2)}\right)^{\frac{d+z-2}{d+z-1}} \sqrt{\frac{d+z-1}{d+z-3}}, \qquad S_{\text{th}} = \frac{1}{4G\rho_H^{d-1}}$$

- Late time saturation $t \sim \frac{\ell}{2}$

$$\Delta S = \frac{L^{D-1} \ell}{4G\rho_H^{d-1}} + \cdots \; .$$

Applications of entanglement entropy

The entanglement entropy can be used as an order parameter to study several aspects of quantum many-body physics.

It may characterize different phases and phase transitions and in particular quantum phase transitions.

It exhibits different scaling behaviours in the process of the thermalization after a global quench.

Part two

Entanglement thermodynamics

Thermodynamics provides a useful tool to study a system when it is in the thermal equilibrium. In this limit the physics may be described in terms of few macroscopic quantities such as energy, temperaure, pressure, entropy.

There are also laws of thermodynamics which describe how these quantities behave under various conditions. In particular the first law of thermodynamics which is energy conservation, tells us how the entropy change as one changes the energy of the system.

There are several interesting phenomena which occur when the system is far from thermal equilibrium.

The entanglement entropy may provide a useful quantity to study excited quantum systems which are far from thermal equilibrium. For a generic quantum system it is difficult to compute the entanglement entropy. Nevertheless, at least, for those quantum systems which have holographic descriptions, one may use the holographic entanglement entropy to explore the behavior of the system.

Another quantity which can be always defined is the energy (or energy density) of the system. It is then natural to pose the question whether there is a relation between the entanglement entropy of an excited state and its energy.

For sufficiently small subsystem, the entanglement entropy is proportional to the energy of the subsystem. The proportionality constant is indeed given by the size of the entangling region.

J. Bhattacharya, M. Nozaki, T. Takayanagi and T. Ugajin, "Thermodynamical Property of Entanglement Entropy for Excited States," Phys. Rev. Lett. **110**, 091602 (2013) [arXiv:1212.1164 [hep-th]].

Recall

Gravity on an asymptotically locally AdS provides a holographic description for a strongly coupled quantum field with a UV fixed point.

The information of quantum state in the dual field theory is encoded in the bulk geometry. In particular the AdS geometry is dual to the ground state of the dual conformal field theory.

Exciting the dual conformal field theory from the ground state to an excited state holographically corresponds to modifying the bulk geometry from AdS solution to a general asymptotically locally AdS solution.

First law

The aim is to compute the entanglement entropy of an excited state for the case where the entangling region is sufficiently small.

Since the entanglement entropy for a small subsystem would probe the UV region of the theory, from holography point of view one only needs to know the asymptotic behavior of the bulk geometry.

On the other hand it is known that the most general form of the asymptotically locally AdS may be written in terms of the Fefferman-Graham coordinates as follows

$$ds^2_{d+1} = \frac{R^2}{r^2}\Big(dr^2 + g_{\mu\nu}dx^\mu dx^\nu\Big),$$

where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x,r)$ with

$$h_{\mu\nu}(x,r) = h^{(0)}_{\mu\nu}(x) + h^{(2)}_{\mu\nu}(x)r^2 + \cdots + r^d\left(h^{(d)}_{\mu\nu}(x) + \hat{h}^{(d)}_{\mu\nu}(x)\log r\right) + \cdots$$

The log term is present for even d. The information about the excited state (or the bulk geometry) is encoded in the function $h_{\mu\nu}(x,r)$.

Let's compute the holographic entanglement entropy for a strip in an AdS geometry. A $d+1$ dimensional AdS solution in the Poincaré coordinates may be written as follows

$$ds^2 = \frac{R^2}{r^2}(dr^2 + \eta_{\mu\nu}dx^\mu dx^\nu), \qquad \mu, \nu = 0, 1, \cdots, d-1.$$

Let us consider an entangling region in the shape of a strip with the width of ℓ given by

$$-\frac{\ell}{2} \le x_1 \le \frac{\ell}{2}, \qquad 0 \le x_i \le L, \qquad i = 2, \cdots, d-1.$$

The holographic entanglement entropy may be computed by minimizing a codimension two hypersurface in the bulk geometry whose intersection with the boundary coincides with the above strip.

Assuming that the bulk extension of the surface to be parameterized by $x_1 = x(r)$, the corresponding area is given by

$$A_0 = R^{d-1} L^{d-2} \int dr \frac{\sqrt{1 + x'^2}}{r^{d-1}}.$$

By making use of the standard procedure one may minimize the area to get

$$\ell = 2 \int_0^{\tilde{r}_t} dr \frac{(r/\tilde{r}_t)^{d-1}}{\sqrt{1 - (r/\tilde{r}_t)^{2(d-1)}}},$$

$$S_E^{(0)}(\tilde{r}_t) = \frac{R^{d-1} L^{d-2}}{4 G_N} \int_{\epsilon}^{\tilde{r}_t} \frac{dr}{r^{d-1}\sqrt{1 - (r/\tilde{r}_t)^{2(d-1)}}},$$

where $\tilde{r}_t$ is turning point and ϵ is a UV cut off. Thus one gets

$$S_E^{(0)} = \frac{L^{d-2} R^{d-1}}{4(d-2) G_N} \left[\frac{1}{\epsilon^{d-2}} - 2^{d-2} \pi^{(d-1)/2} \left(\frac{\Gamma\left(\frac{d}{2d-2}\right)}{\Gamma\left(\frac{1}{2d-2}\right)} \right)^{d-1} \frac{1}{\ell^{d-2}} \right],$$

Let's now compute the holographic entanglement entropy for a strip in an AdS black hole geometry.

$$dS^2 = \frac{L^2}{r^2}\left(-f(r)dt^2 + g(r)dr^2 + dx_1^2 + dx_{d-2}^2\right), \qquad f(r) = g(r)^{-1} = 1 - mr^d$$

For the strip, the induced metric on this hypersurface

$$dS^2_{\text{ind}} = \frac{R^2}{r^2}\left[\left(g(r) + x'^2\right)dr^2 + d\vec{x}^2\right].$$

Therefore the area A reads

$$A = L^{d-2}R^{d-1}\int dr \frac{\sqrt{g + x'^2}}{r^{d-1}},$$

$$\frac{\ell}{2} = \int_0^{r_t} dr \frac{\sqrt{g(r)}\left(\frac{r}{r_t}\right)^{d-1}}{\sqrt{1 - \left(\frac{r}{r_t}\right)^{2(d-1)}}}, \qquad S = \frac{L^{d-2}R^{d-1}}{2G_N}\int_\epsilon^{r_t} \frac{\sqrt{g(r)}dr}{r^{d-1}\sqrt{1 - \left(\frac{r}{r_t}\right)^{2(d-1)}}}$$

In the limit of $ml^d \ll 1$ the change of the entanglement entropy is

$$\Delta S = S - S_0 = \frac{L^{d-2}R^{d-1}}{32(d+1)G_N}\frac{ml^2}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2(d-1)})^2\Gamma(\frac{1}{d-1})}{\Gamma(\frac{d}{2(d-1)})^2\Gamma(\frac{1}{2}+\frac{1}{d-1})}$$

On the other hand since the change of the energy is

$$\Delta E = \frac{(d-1)L^{d-2}R^{d-1}ml}{16\pi G_N}$$

Therefore one finds

$$\Delta S = c_0 \ell \Delta E$$

which can be recast to the following first law of the entanglement entropy

$$\Delta E = T_E \Delta S$$

The entanglement temperature is

$$T_E \sim \frac{1}{\ell}$$

General features

Consider a deformation of the AdS geometry which in turn corresponds to dealing with an excited state in the dual field theory.

The aim is to compute the entanglement entropy of the strip for an excited state when the width of strip is sufficiently small in which the only UV regime of the system will be probed.

Using the notation of the Fefferman-Graham coordinates, we assume that $h^{(n)}_{\mu\nu}\ell^n \ll 1$. Note that in this limit, practically one needs to compute the minimal surface up to order of $\mathcal{O}(h)$.

For the above strip the induced metric in the Fefferman-Graham coordinates is

$$ds^2 = \frac{R^2}{r^2}\left((1 + g_{11}x'^2)dr^2 + 2g_{1i}x'drdx^i + g_{ij}dx^i dx^j\right).$$

Therefore to find the holographic entanglement entropy one needs to minimize the following area

$$A = R^{d-1}\int d^{d-2}xdr\ \frac{\sqrt{g(r)\ (1 + G(r)\ x'^2)}}{r^{d-1}}$$

where $g(r) = \det(g_{ij})$ and $G(r) = g_{11} - g_{1i}g_{ij}^{-1}g_{j1}$.

1. Consider the case where the solution is static.

2. To find analytic expressions for our results we will assume that the components of the asymptotic metric are independent of x_1, the direction the width of strip is exended.

With these assumption the equation of motion of x leads to a constant of motion

$$\left(\frac{R}{r}\right)^{d-1} \frac{\sqrt{g(r)}\, G(r)\, x'}{\sqrt{1+G(r)\, x'^2}} = \text{const} = c,$$

so that

$$x' = \frac{c}{\sqrt{G(r)\left[g(r)\, G(r)\, (\frac{R}{r})^{2(d-1)} - c^2\right]}}.$$

The constant c may be found in terms of the turning point where x' diverges.

Denoting by r_t the turning point, one finds

$$c^2 = g(r_t)\; G(r_t)\; \left(\frac{R}{r_t}\right)^{2(d-1)}$$

It is then straightforward to find the entanglement entropy and the width of the strip as follows

$$S_E = \frac{1}{4G_N}\int_0^{r_t} d^{d-2}x dr\, \left(\frac{R}{r}\right)^{2(d-1)} \sqrt{\frac{g(r)^2 G(r)}{g(r)\; G(r)\; (\frac{R}{r})^{2(d-1)} -\; c^2}}$$

$$\ell = 2\int_0^{r_t} dr\, \frac{c}{\sqrt{G(r)\Big[g(r)\; G(r)\; (\frac{R}{r})^{2(d-1)} -\; c^2\Big]}}$$

To evaluate the above expressions we note that at leading order one has

$$g(r) = 1 + \mathrm{Tr}(h_{ab}) - h_{11} + \mathcal{O}(h^2), \qquad G(r) = 1 + h_{11} + \mathcal{O}(h^2),$$

where $a, b = 1, 2, \cdots, d-1$. So that $g(r)G(r) = 1 + \mathrm{Tr}(h_{ab}) + \mathcal{O}(h^2)$.

In what follows in order to simplify the expressions, it is found useful to define the following parameters

$$\gamma(r) = \mathrm{Tr}(h_{ab}), \quad \beta(r) = h_{11}, \quad f(r, r_t) = \sqrt{1 - \left(\frac{r}{r_t}\right)^{2(d-1)}}.$$

In this notation at the first order in h one arrives at

$$\ell = \int_0^{r_t} \frac{(r/r_t)^{d-1}}{f(r, r_t)} \left[2 + \frac{\gamma(r_t) - \gamma(r)}{f^2(r, r_t)} - \beta(r)\right] dr$$

We are interested in the change of the entanglement entropy caused by the change of the state. We keep the entangling surface fixed.

Since ℓ is kept fixed while the geometry is deformed the turning point should also be changed. Indeed assuming $r_t = \tilde{r}_t + \delta r_t$ with $\tilde{r}_t$ being the turning point for the pure AdS case, one finds

$$\delta r_t = -\frac{1}{2a_d}\int_0^{\tilde{r}_t} \frac{(r/\tilde{r}_t)^{d-1}}{f(r,\tilde{r}_t)}\left[\frac{\gamma(\tilde{r}_t)-\gamma(r)}{f^2(r,\tilde{r}_t)} - \beta(r)\right]dr$$

where

$$a_d = \int_0^1 \frac{\xi^{d-1}}{\sqrt{1-\xi^{2(d-1)}}}d\xi,$$

Moreover the width of the strip ℓ is the same as that in pure AdS geometry which is $\ell = 2\tilde{r}_t a_d$.

It is straightforward to compute the entanglement entropy up to order of $\mathcal{O}(h)$. In fact expanding the expression of the entanglement entropy one finds

$$S_E = S_E^{(0)}(\tilde{r}_t) + \frac{R^{d-1}}{8G_N} \int_0^{\tilde{r}_t} dr\; d^{d-2}x \, \frac{\gamma(r) - f^2(r, \tilde{r}_t)\beta(r)}{r^{d-1} f(r, \tilde{r}_t)},$$

where $S_E^{(0)}(\tilde{r}_t)$ is the holographic entanglement entropy for the strip in a pure AdS$_{d+1}$ geometry.

By making use of the Fefferman-Graham expansion for the asymptotic form of the metric one arrives at

$$\Delta S_E = \frac{R^{d-1}}{8G_N}\int_0^{\tilde r_t} dr\left(\Gamma^{(0)} + \Gamma^{(2)} r^2 + \cdots + \Gamma^{(d)} r^d + \tilde\Gamma^{(d)} r^d \ln r\right),$$

where the change of the entanglement entropy is defined by

$$\Delta S_E = S_E - S_E^{(0)}(\tilde r_t)$$

also

$$\Gamma^{(n)} = \frac{\int d^{d-2}x\ \mathrm{Tr}(h_{ab}^{(n)})}{r^{d-1} f(r,\tilde r_t)} - \frac{f(r,\tilde r_t)}{r^{d-1}}\int d^{d-2}x\ h_{11}^{(n)}$$

$$\tilde\Gamma^{(d)} = \frac{\int d^{d-2}x\ \mathrm{Tr}(\tilde h_{ab}^{(d)})}{r^{d-1} f(r,\tilde r_t)} - \frac{f(r,\tilde r_t)}{r^{d-1}}\int d^{d-2}x\ \tilde h_{11}^{(d)}.$$

Using this expansion it is straightforward to perform the integration over r. Indeed for $d > 2$ one finds

$$\int_\epsilon^{\tilde{r}_t} dr\, \Gamma^{(n)} r^n = \frac{1}{(d-2-n)\epsilon^{d-2-n}} \int d^{d-2}x \left(\mathrm{Tr}(h^{(n)}_{ab}) - h^{(n)}_{11} \right)$$
$$- \frac{F(d-1, d-1-n)}{\tilde{r}_t^{d-2-n}} \int d^{d-2}x \left(\mathrm{Tr}(h^{(n)}_{ab}) - \frac{d-1}{n+1} h^{(n)}_{11} \right)$$
$$\equiv \frac{1}{(d-2-n)\epsilon^{d-2-n}}\, N^{(n)} + \frac{1}{\tilde{r}_t^{d-2-n}}\, M^{(n)},$$

where ϵ is a UV cut off, and

$$F(m,n) = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1-n}{2m}, \frac{2m+1-n}{2m}, 1\right)}{n-1},$$

with ${}_2F_1$ being the hypergeometric function.

Note that for even d and $n = d - 2$ one finds just a logarithmic divergence as $N^{(d-2)}$ $\ln \frac{\epsilon}{\tilde{r}_t}$ while for odd d and $n = d - 1$ the result is finite and is given by $M^{(d-1)}\, \tilde{r}_t$.

Forarbitrary d for $n = d$ it leads to a finite term given by $\tilde{r}_t^2 M^{(d)}$. More precisely, using the fact that in general at leading order $\mathrm{Tr}(h_{\mu\nu}^{(d)}) = \mathcal{A}$ with $\mathcal{A}$ being the trace anomaly one finds

$$\int_0^{\tilde{r}_t} dr\, \Gamma^{(d)} r^d = -F(d-1,-1)\, \tilde{r}_t^2 \int d^{d-2}x \left(h_{tt}^{(d)} + \mathcal{A} - \frac{d-1}{d+1} h_{11}^{(d)} \right).$$

Note that for odd d the anomaly term is zero. One should add that when d is an even number we have another term coming from $\tilde{\Gamma}^{(d)}$ which can similarly be calculated leading to an $\ln \tilde{r}_t$ contribution to the entanglement entropy.

Setting $\ell = 2\tilde{r}_t a_d$, one can find the variation of the entanglement entropy, ΔS_{EE}, as a function of ℓ. For odd d

$$
\begin{aligned}
\Delta S_E \;=\; & \frac{R^{d-1}}{8G_N} \sum_{n<d-2} \left(\frac{1}{(d-2-n)\epsilon^{d-2-n}}\, N^{(n)} + \frac{(2a_d)^{(d-2-n)}}{\ell^{d-2-n}}\, M^{(n)} \right) \\
& + \frac{R^{d-1} M^{(d-1)}}{16 G_N a_d}\, \ell - \frac{R^{d-1} F(d-1,-1)}{32 a_d^2 G_N}\, \ell^2 \int d^{d-2}x \left(h_{tt}^{(d)} - \frac{d-1}{d+1} h_{11}^{(d)} \right)
\end{aligned}
$$

while for even d one gets

$$
\begin{aligned}
\Delta S_E \;=\; & \frac{R^{d-1}}{8G_N} \sum_{n<d-2} \left(\frac{1}{(d-2-n)\epsilon^{d-2-n}}\, N^{(n)} + \frac{(2a_d)^{d-2-n)}}{\ell^{d-2-n}}\, M^{(n)} \right) \\
& + \frac{R^{d-1} N^{(d-2)}}{8G_N} \ln \frac{2\epsilon a_d}{\ell} \\
& - \frac{R^{d-1} F(d-1,-1)}{32 a_d^2 G_N}\, \ell^2 \int d^{d-2}x \left(h_{tt}^{(d)} + \mathcal{A} - \frac{d-1}{d+1} h_{11}^{(d)} \right) \\
& - \frac{R^{d-1} F(d-1,-1)}{32 a_d^2 G_N}\, \ell^2 \ln \frac{\ell}{2a_d} \int d^{d-2}x \left(\tilde{h}_{tt}^{(d)} - \frac{d-1}{d+1} \tilde{h}_{11}^{(d)} \right)
\end{aligned}
$$

Here we have used the fact that $\mathrm{Tr}(\tilde{h}_{\mu\nu}^{(d)}) = 0$.

When one excites the ground state to an excited state, the energy of the system is increased and generally one gets non-zero expectation value for the energy momentum tensor.

$$\langle T_{\mu\nu} \rangle = \frac{dR^{d-1}}{16\pi G_N} \, h^{(d)}_{\mu\nu}$$

The extra non-trivial contribution to the entanglement entropy is coming from expectation value of the energy-momentum tensor which does depend on the excited state we are considering.

S. de Haro, S. N. Solodukhin and K. Skenderis, "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence," Commun. Math. Phys. **217**, 595 (2001) [hep-th/0002230].

More precisely one finds

$$\Delta S_E^{\text{finite}} = \sum_n (\cdots)\frac{M^{(n)}}{\ell^{d-2n}} - \frac{\pi F(d-1,-1)\ell}{2a_d^2 d}\left(\Delta E - \frac{d-1}{d+1}\int \Delta P_x dV_{d-1} + \frac{dR^{d-1}}{16\pi G_N}\int \mathcal{A} dV_{d-1}\right) + \cdots,$$

where $(\cdots)$ stands for some numerical factors and $dV_{d-1} = \ell d^{d-2}x$. Moreover the energy and entanglement pressure are defined by

$$\Delta E = \int dV_{d-1}\langle T_{tt}\rangle, \qquad \Delta P_x = \langle T_{11}\rangle.$$

For the case of $h^{(0)}_{\mu\nu} = 0$ where one has

$$h_{\mu\nu}(x,r) = h^{(d)}_{\mu\nu}(x)\, r^d$$

the boundary is flat the anomaly term is zero and therefore one gets

$$\Delta S_E = \frac{\pi\ell}{4d}\frac{C_1}{C_0^2}\left(\Delta E - \frac{d-1}{d+1}\int dV_{d-1}\Delta p_x\right),$$

where

$$C_0 = \sqrt{\pi}\,\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}, \qquad C_1 = \sqrt{\pi}\,\frac{\Gamma\left(\frac{d}{d-1}\right)}{\Gamma\left(\frac{d+1}{2(d-1)}\right)},$$

One may define entanglement temperature in terms of the width of the strip. In the present case the corresponding temperature may be given by

$$T_E = \frac{4dC_0^2}{\pi C_1} \frac{1}{\ell}$$

Assuming that $h_{\mu\nu}^{(d)}$ to be constant one gets

$$\Delta E = T_E \Delta S_E + \frac{d-1}{d+1} V_{d-1} \Delta p_x$$

where V_{d-1} is the volume of the entangling region.

Due to its similarity with the first law of thermodynamics we would like to consider this expression as the first law of entanglement thermodynamics.

General form

So far we have considered the static case where the corresponding background geometry was time independent.

It is, however, possible to show that the final results also hold for time dependent cases.

As long as we are interested in a sufficiently small subsystem we could still use the static solution leading to the same result for the first law.

Consider a time dependent excitation state above a vaccum solution. From the bulk point of view it corresponds a time dependent deviation from AdS solution.

There are several sources which contribute to the change of the holographic entanglement entropy. The change may be caused by the change of the turning point, the change of the solution and the change of the metric.

The interesting point is that at leading order which is what we are interested in the change of entanglement entropy is completey given by the change of metric

$$\Delta S_E = \frac{1}{4G_N}\int d^{d-1}x\sqrt{\det(g_{\rm in}^{(0)})}(g_{{\rm in}\ ab}^{(0)})^{-1}g_{{\rm in}\ ab}^{(1)},$$

where $g_{\rm in}^{(0)}$ and $g_{\rm in}^{(1)}$ are the induced metrics on the codimension two hypersurface in the bulk for the cases of AdS geometry and the perturbation above it, respectively.

The result is the same as that we considered in the previous section. Therefore the first law we have inroduced may also be applyed for the time dependent case.

M. Nozaki, T. Numasawa and T. Takayanagi, "Holographic Local Quenches and Entanglement Density," arXiv:1302.5703 [hep-th].

Other laws of entanglement thermodynaics

Based on the holographic description of the entanglement entropy and for explicit examples we have found a relation between entanglement entropy, energy and entanglement pressure which using the similarity with the thermodynamics could be thought of as the first law of entanglement thermodynamics.

It is then natural to pose the question whether there are other laws similar to what we have in the thermodynamics.

Second law

There is a natural statement for the second law of entanglement thermodynamics: the strong subadditivity.

According to the strong subadditivity for any given two subsystems A and B one has

$$S_{E(A)} + S_{E(B)} \leq S_{E(A\cup B)} + S_{E(A\cap B)}.$$

It is worth noting that although the entanglement entropy is divergent due to UV effects, the divergent parts of the entanglement entropy drop from both sides. In fact this inequality is also satisfied by the finite part of the entanglement entropy.

So far our suggestions and statements about the laws of entanglement thermodynamics were based on rigorous computations.

To proceed for other possible laws we note that although we will use an explicit example to explore them.

The most important part of our study is the definition of the entanglement temperature.

From dimensional analysis and also from our experiences in thermodynamics and hydrodynamics it is natural to consider the inverse of the typical size of the entangling region as the temperature.

But there is a non-universal numerical factor in its definition!

As long as we are considering entangling regions with a fixed shape the numerical factor is universal.

Zeroth law

Apart from this ambiguity, in what follows for a fixed shape we suggest a statement which could be considered as the zeroth law of entanglement thermodynamics.

Consider two entangling regions given by two strips with the width of ℓ_1 and ℓ_2, respectively.

When they joined together we get another strip whose width at most could be $\ell_3 = \ell_1 + \ell_2$, Or

$$\ell_1 + \ell_2 \geq \ell_3$$

Using the definition of the entanglement temperatures before and after joining one gets

$$\frac{1}{T_{1E}} + \frac{1}{T_{2E}} \geq \frac{1}{T_{3E}}$$

It is easy to argue that such a relation could also be satisfied when the entangling regions are spheres.

For a special case where the system is isotropic one can find this relation from strong subadditivity.

$$\Delta S_{1E} + \Delta S_{2E} \geq \Delta S_{3E}$$

which results, taking into account $T_E \Delta S_E \sim \Delta E$

$$\frac{V^{(1)}_{d-1}\mathcal{E}}{T_{1E}} + \frac{V^{(2)}_{d-1}\mathcal{E}}{T_{2E}} \geq \frac{V^{(3)}_{d-1}\mathcal{E}}{T_{3E}}$$

Then (for strip)

$$\frac{1}{T^2_{1E}} + \frac{1}{T^2_{2E}} \geq \frac{1}{T^2_{3E}}$$

Therefore one finds

$$\left(\frac{1}{T_{1E}} + \frac{1}{T_{2E}}\right)^2 \geq \frac{1}{T^2_{1E}} + \frac{1}{T^2_{2E}} \geq \left(\frac{1}{T_{3E}}\right)^2$$

Third law

Let us now proceed to introduce the third law of entanglement thermodynamics.

Consider the finite part of the entanglement entropy of a strip for an excited state up to order of $\mathcal{O}(T_E^{-2})$

$$
\begin{aligned}
S_E^{\text{finite}} = & \frac{R^{d-1}}{8G_N}\left[(\tilde{B}_0 L^{d-2} + B_0 M^{(0)})T_E^{d-2} + \sum_{n=1} B_n M^{(n)} T_E^{d-2-n}\right] \\
& + \frac{1}{T_E}\left(\Delta E + \frac{d-1}{d+1}\Delta P_x \Delta V_{d-1}\right)
\end{aligned}
$$

where $\tilde{B}_0, B_n$ are numerical factors.

$$S_E^{\text{finite}} \sim T_E^{d-2} \quad \text{for large } T_E$$

The finite part of entanglement entropy goes to infinity for sufficiently higher entanglement temperature.

Due to a natural UV cut off in the theory there is a natural cut off for temperature preventing to get infinite entanglement entropy.

Note that as we increase the temperature, the dominant divergent parts comes from the ground state which corresponds to the AdS geometry. it is then possible to argue that the above statement is also valid for other shape of the entangling region.

Laws of entanglement thermodynamics

- Zeroth law: The entanglement temperature is proportional to the inverse of the typical size of the entangling region and for two subsystem A and B one has

$$\frac{1}{T_{(A)E}} + \frac{1}{T_{(B)E}} \geq \frac{1}{T_{(A\cup B)E}}.$$

- First law: There is a relation between the energy of the system and the entanglement entropy as follows

$$\Delta E = T_E \Delta S_E + \frac{d-1}{d+1} V_{d-1} \Delta P_\perp,$$

where $\Delta P_\perp$ is the entanglement pressure normal to the entangling surface.

- Second law: Entanglement entropy enjoys strong subadditivity

$$S_{E(A)} + S_{E(B)} \leq S_{E(A \cup B)} + S_{E(A \cap B)}$$

- Third law: There is an upper bound on the entanglement temperature preventing to have an infinite entanglement entropy.

An explicit example

The AdS Schwarzschild background

$$ds^2 = \frac{R^2}{\rho^2}\left(-f(\rho)dt^2 + \frac{d\rho^2}{f(\rho)} + \sum_{i=1}^{d-1} dx_i^2\right), \qquad f(\rho) = 1 - \left(\frac{\rho}{\rho_H}\right)^d$$

where ρ_H is the radius of horizon. By making use of the coordinate transformation $\frac{dz}{z} = \frac{d\rho}{\rho f^{1/2}}$, one may recase the metric to the Fefferman-Graham coordites as follows

$$ds^2 = \frac{R^2}{r^2}(dr^2 + g_{\mu\nu}dx^\mu dx^\nu),$$

whose asymptotic behvior of the metric components are

$$g_{tt} = -1 + h_{tt}^{(d)} r^d = -1 + \frac{4(d-1)}{d}\rho_H^d r^d, \qquad g_{aa} = 1 + h_{aa}^{(d)} r^d = 1 + \frac{4}{d}\rho_H^d r^d$$

So $\Delta E = \frac{4(d-1)}{d}\rho_H^d V_{d-1}$ and $\Delta P_x = \frac{4}{d}\rho_H^d$. From first law one finds

$$T_E \Delta S_E = \frac{4(d-1)}{d+1}\rho_H^d V_{d-1}.$$

Modular Hamiltonian

Consider a general quantum system. For any satet in the system, the state of a subsystem A is decsribed by reduced density matrix

$$\rho_A = \mathrm{Tr}_{A_c}(\rho_{\mathrm{total}})$$

where ρ_{total} is the density matrix of the system and A_c is the complement of A.

The entanglement entropy is defined by the von Neumann entropy

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A$$

Since the reduced density matrix is both hermition and positive (semi) definite, it may be expressed sa

$$\rho_A = \frac{e^{-H_A}}{\mathrm{Tr}(e^{-H_A})}, \qquad \mathrm{Tr}(\rho_A) = 1$$

H_A is modular Hamiltonian.

Consider any infinitesimal variation to the state of the system. At first order one gets

$$\begin{aligned}\delta S_A &= -\mathrm{Tr}(\delta\rho_A \log \rho_A) - \mathrm{Tr}(\rho_A \rho_A^{-1} \delta\rho_A) \\ &= \mathrm{Tr}(\delta\rho_A H_A) - \mathrm{Tr}(\delta\rho_A)\end{aligned}$$

Therefore the variation of entanglement entropy satisfies

$$\delta S_A = \delta\langle H_A\rangle$$

where H_A is associated with the original unperturbed state.

For a general quantum field theory, general state and general entangling region, the modular Hamiltonian it not known.

T. Faulkner, M. Guica, T. Hartman, R. C. Myers and M. Van Raamsdonk, "Gravitation from Entanglement in Holographic CFTs," arXiv:1312.7856 [hep-th].

For a conformal field theory in its vaccum state $\rho_{\text{total}} = |0\rangle\langle 0|$ in d-dimensional Minkowski space and an entangling region in a form of a ball, the modular Hamiltonian has a simple form.

Consider a ball with radius R_0 on a time slice $t = t_0$ and centered at $x^i = x_0^i$ one has

$$H_{\text{Ball}} = 2\pi \int_{\text{Ball}} d^{d-1}x \, \frac{R_0^2 - |\vec{x} - \vec{x}_0|^2}{2R_0} T_{tt}(t_0, \vec{x})$$

where $T_{\mu\nu}$ is stress tensor.

P. D. Hislop and R. Longo, "Modular Structure of the Local Algebras Associated With the Free Massless Scalar Field Theory," Commun. Math. Phys. 84, 71 (1982).

H. Casini, M. Huerta and R. C. Myers, "Towards a derivation of holographic entanglement entropy," JHEP **1105**, 036 (2011) [arXiv:1102.0440 [hep-th]].

Therefore, starting from the vaccum state for any CFT and a ball-shaped entangling region, the first law reduces to

$$\delta S_B = \delta E_B$$

where

$$E_B = 2\pi \int_{\text{Ball}} d^{d-1}x \;\; \frac{R_0^2 - |\vec{x} - \vec{x}_0|^2}{2R_0} \langle T_{tt}(t_0, \vec{x}) \rangle$$

For a CFT theory which has gravitational description, both δS and δE can be computed from gravity side. Then the first law is a constraint on the small perturbations around the vacuum AdS solution.

- For small perturbation around the vacuum solution satisfies linear equations of motion, the first law would hold.

- A small perturbation which satisfies first law, will obey linear equations of motion.

Application

First law applied to infinitesimal ball shaped enatangling regions may be used to compute holographic stress tensor and constrains the asymptotic brhaviour of the metric.

Summary

1. Entanglement entropy is a good order parameter

2. There is very nice simple holographic description of entanglement entropy

3. One may define a framework for entanglement entropy such as thermodynamics

4. First law of entanglement entropy may provide an alternative way for holographic renormalization