17-1-2014

Puri: Advanced School

Lecture 2

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Arguments that $\tilde{O}$ do not exist

a) strong subadditivity
b). locality with complementarity
c). lack of a left-inverse
d) $N_a \neq 0$ argument.

In Lecture 3, I will present a resolution to each of these arguments, but I feel these arguments are deep and interesting, so today I will present them without giving the solutions

1) strong subadditivity paradox (Mathur, AMPS)

Before I describe the strong subadditivity paradox, let me describe the "Page curve".

The claim is that if one take a hilbert space with dim(2^N). and writes it as a direct product

$$\begin{array}{ccccc} H_N & = & H_M & \otimes & H_{N-M} \\ \downarrow & & \downarrow & & \downarrow \\ 2^N & & 2^M & & 2^{N-M}. \end{array}$$

then, consider a generic pure state in H_N

$$|\psi\rangle_N = \sum_{i,j} |\psi_i\rangle_M \, |\psi_j\rangle_{N-m} \, a_{ij}$$

then, for

$$M \leq \frac{N}{2},$$

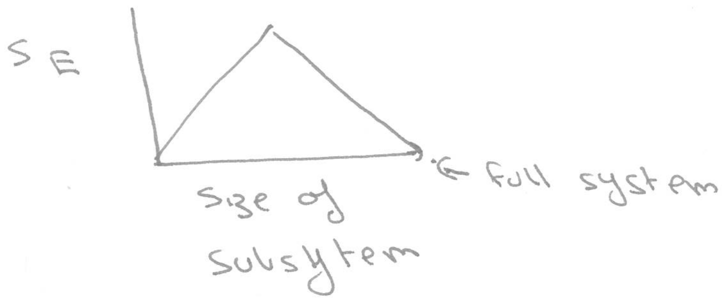
we have,

$$S_M = -\mathrm{tr}(\rho_M \ln \rho_M) = M \ln 2.$$

For $M \geq \frac{N}{2}$

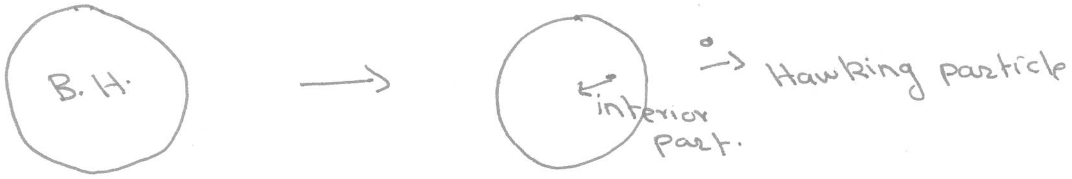
$$S_M = (N-M)\ln 2$$

In other words, if we take a pure state and divide it into two parts, the entanglement entropy goes like:



2) Now let us return to black holes.

Consider an evaporating B.H.



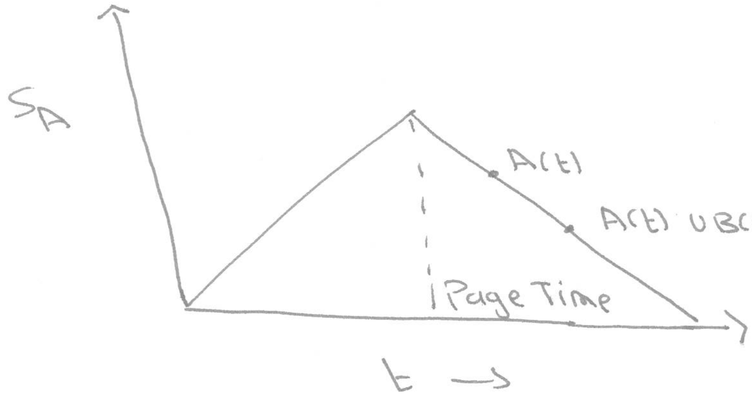
we have 3-subsystems

A = collection of old radiation

B = particle being emitted at some point

C = pair inside the B.H.

3) Now, if we apply the Page analysis to A, we find that



Now imagine a B.H. after the Page Time.
Then, since A + B will be the new A

$$A(t+\delta t) \approx A(t) \cup B(t)$$

so

$$S_{AB} < S_A$$

But

$$S_{BC} = 0$$

for a smooth horizon.

This is the statement that the emitted particle is maximally entangled with the particle "inside"

In our analysis, we saw how modes outside must be entangled with the modes inside.

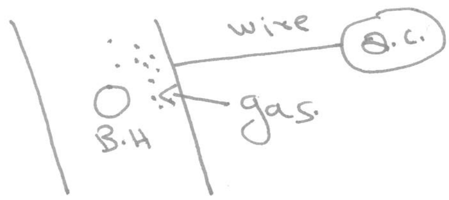
However, also

$$S_B = S_C > 0$$

This is in <u>contradiction</u> with the strong subadditivity of entropy:

$$S_{AB} + S_{BC} \geq S_A + S_C.$$

4) This whole story has not been made very precise in terms of WKB correlators. However to make this precise, we can imagine the following



we turn on absorbing boundary conditions on the boundary, or else couple the CFT to another system with a <u>much greater</u> entropy.

eg.

$$S \to S + \int Tr(F^2) . J(x) \, dx$$

where $J(x)$ is an operator in the auxiliary theory.

Now, we derived that the gas surrounding the B.H. was entangled with something in the interior so <u>that together</u>, they look like a pure state:

$$|\Omega\rangle_{H.H} = e^{\int e^{-\beta\omega/2} a^\dagger_\omega \tilde{a}^\dagger_\omega} |\Omega_{schwarz}\rangle$$

$|\Omega\rangle_{CFT}$

Now, once again as the energy slowly leaks out, we run into the strong subadditivity paradox

5) Let me now give some other arguments, which also work for a big B. H.
Think of the operator

$$\tilde{O}_\omega^+$$

Now

$$\tilde{O}_\omega^+$$

acts like O_ω, so it <u>lowers</u> the energy.

$$[H, O_\omega^+] = -\omega O_\omega^+$$

But we also have:

$$\left[\frac{O_\omega}{\sqrt{G_\omega}}, \frac{O_{\omega'}^+}{\sqrt{G_\omega}}\right] = \delta(\omega-\omega').$$

We can smooth this out a little by defining

$$\alpha(f) = \int \frac{O_\omega}{\sqrt{G_\omega}} f(\omega)\, d\omega$$

$$\alpha^+(f) = \int \frac{O_\omega^+}{\sqrt{G_\omega}} f^*(\omega)\, d\omega.$$

$$[\alpha(f), \alpha^+(f)] = \int |f(\omega)|^2 d\omega = 1$$

for properly normalized f.

we can, nevertheless, take F to be very sharply peaked about some $\bar{\omega}$ so that

$$[H, \alpha^{+}(F)] \approx \bar{\omega}\, \alpha^{+}(F)$$

6) But, now $\alpha^{+}(F)$ must have a left inverse:

$$\alpha_F \alpha_F^{+} = \alpha_F^{+} \alpha_F + 1$$

or

$$\gamma_F \alpha_F^{+} = 1$$

with

$$\gamma_F = \cancel{\alpha_F} \, (1 + \alpha_F^{+} \alpha_F)^{-1} \alpha_F .$$

7) But, the CFT has a density of states that grows <u>monotonically</u> with energy. A map from higher to lower energy cannot have a left inverse!

So α^{+} cannot exist in the CFT.

8) Finally, we come to the

$$N_\alpha \neq 0$$

paradox.

Roughly speaking, the argument is as follows. If we consider some state the condition that the infalling observer sees the vacuum is that

$$\alpha_1 |\psi\rangle = \left(a_\omega - e^{-\beta\omega/2} \tilde{a}^\dagger_\omega\right) |\psi\rangle = 0.$$

$$\alpha_2 |\psi\rangle = \left(\tilde{a}_\omega - e^{-\beta\omega/2} a^\dagger_\omega\right) |\psi\rangle = 0$$

But, the CFT Hamiltonian looks like the <u>Rindler</u> <u>Hamiltonian</u>. So, it does not look like there is any <u>energetic</u> <u>reason</u> for this. So, in a generic microstate, we will expect that this condition will not be satisfied.

9) The original phrasing of the paradox by Marolf and Polchinski is as follows. We know the N_α vacuum has a thermal distribution of N_a, and N_a eigenstates differ at $O(1)$ from the N_α vacuum

But

$$[H, N_a] = 0 + \frac{1}{N}$$

and so, at infinite N, one can diagonalize H and N_a simultaneously.

[Notational point: Here we use "a" for what is commonly called "l" and α for what is called "a"]

So, N_a eigenstates are also energy eigenstates at infinite N

Claim: At $\frac{1}{N}$ also, one can write N_a eigenstates in terms of energy eigenstates with some bounded spread.

Now, to compute

$$\langle N_\alpha \rangle = \mathrm{Tr}_{E_0 - \Delta < E < E_0 + \Delta} N_\alpha$$

we could instead compute

$$\mathrm{Tr}_{N_\alpha \text{ eigenstates}} N_\alpha > 0$$

because N_α is positive