Brownian motion of a heavy particle in strongly coupled anisotropic medium via holography

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[D. Giataganas, H. S., arXiv:1310.6725, 1312.7474]

Plan

- Motivation
- Moving quark in a 4D Field Theory & Langevin equation
- Trailing string in 5D gravity (Fluctuations, broadening parameters)
- Gravity/FT agreement and Einstein relation
- Anisotropic asymptotically AdS_5 backgrounds (Top-down, bottom-up)
- Some results for Langevin diffusion coefficients in anisotropic plasmas
- Summary & Conclusion

Motivation

- Strongly Coupled plasma created at RHIC & LHC
  Low Shear viscosity $\frac{\eta}{s} << 1$
  Large jet quenching parameter $\hat{q}$
- QGP created is anisotropic for a short time ($\tau < \tau_{\text{iso}} \sim 1\text{fm}$)
- Heavy quarks are produced in QGP & moving through plasma
- Perturbation theory & Lattice QCD are limited $\Rightarrow$ AdS/CFT
  - Practically (needs super-computers)
  - Intrinsically (Euclidean time$\rightarrow$real phenomena!?)

  $\Rightarrow$ AdS/CFT (At most it gives a ballpark estimate for QCD)
- Trailing open string with an end-point moving in asymp. bdry. (representing the heavy quark)
- General method applicable for any asymp. AdS backgrounds

Moving quark in 4D Field Theory

◇Local Langevin equation with white noise

$$\frac{dp^i}{dt} = -\eta_D^{ij}(p)p_j + \zeta^i(t), \qquad < \zeta^i(t)\zeta^j(t') >= \kappa^{ij}\delta(t-t')$$

Linearizing around a uniform trajectory, $\vec{X}(t) = \vec{v}t + \delta\vec{X}$ in momentum space, $\gamma = (1-v^2)^{-1/2}$

$$\frac{dp_0}{dt} = -\eta_D^L p_0, \qquad p_0 \equiv \gamma M v,$$

$$\gamma^3 M \delta\ddot{X}^L = -\eta_L \delta\dot{X}^L, \qquad < \zeta^L(t)\zeta^L(t') >= \kappa_L \delta(t-t')$$

$$\gamma M \delta\ddot{X}^\perp = -\eta_T \delta\dot{X}^\perp, \qquad < \zeta^\perp(t)\zeta^\perp(t') >= \kappa_T \delta(t-t')$$

$$\eta_T = \gamma M \eta_D^T, \qquad \eta_L = \gamma^3 M \left(\eta_D^L + p\frac{\partial\eta_D^L}{\partial p}\right)_{p=p_0}$$

The aim is: Using AdS/CFT to calculate
friction coefficients η^{ij} and diffusion coefficients κ^{ij}

Moving string (quark) in a 5D bulk (4D bdry) theory

◇Trailing string

The general 5D bulk

$$ds^2 = G_{tt}(u)dt^2 + G_{uu}(u)du^2 + G_{ii}(u)d{X^i}^2, \qquad i = 1, 2, 3$$

Using the static gauge for a moving string in X^1 direction

$$t = \sigma^0, \quad u = \sigma^1, \quad X^1 = v\ t + \xi(u), \quad X^2 = 0, \quad X^3 = 0$$

the induced world-sheet metric $g_{\mu\nu} = G_{MN}\partial_\mu X^M \partial_\nu X^N$

$$g_{\mu\nu} = \begin{pmatrix} G_{tt} + v^2\, G_{11} & G_{11}\, v\, \xi' \\ G_{11}\, v\, \xi' & G_{uu} + \xi'^2\, G_{11} \end{pmatrix}$$

and the Nambu-Guto action

$$S_{NG} = -\frac{1}{2\,\pi\,\alpha'} \int d^2\sigma \sqrt{-g}, \quad \Pi_\xi := \frac{\partial \mathcal{L}}{\partial \xi'} = \text{const.}$$

Moving string (quark) in a 5D bulk (4D bdry) theory

◇Trailing string

$$\xi'^2 = -C^2 \frac{G_{uu}}{G_{tt}\,G_{11}} \frac{G_{tt} + G_{11}\,v^2}{G_{tt}\,G_{11} + C^2}, \qquad C^2 = 4\pi^2\alpha'^2\Pi_\xi^2$$

There is a critical point

$$G_{tt}(u_c) + G_{11}(u_c)\,v^2 = 0, \qquad G_{tt}(u_c)\,G_{11}(u_c) + C^2 = 0$$

which is the horizon for World-sheet metric,

$$g_{\mu\nu} = \begin{pmatrix} G_{tt} + v^2\,G_{11} & G_{11}\,v\,\xi' \\ G_{11}\,v\,\xi' & G_{uu} + \xi'^2\,G_{11} \end{pmatrix}$$

The points above the World-sheet horizon are causally disconnected from the points below the world-sheet horizon.

◇World-Sheet temperature

For a generic 2×2 metric

$$g_{\mu\nu} = \begin{pmatrix} g_{tt}(u) & g_{tu}(u) \\ g_{tu}(u) & g_{uu}(u) \end{pmatrix}$$

one can diagonalize the metric by choosing new coordinates

$$t \to \tau = t + A(u), \qquad A'(u) = \frac{g_{tu}}{g_{tt}},$$

World-sheet metric becomes diagonal

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 \\ 0 & g/g_{tt} \end{pmatrix}, \qquad T_{ws} = \frac{1}{4\pi} \left[\left| \frac{G'^2_{tt} - v^4 \, G'^2_{11}}{G_{tt} \, G_{uu}} \right| \right]^{1/2}_{u=u_c}$$

What is the physical meaning of T_{ws}?!

Effective temperature for a moving thermometer!

Moving string (quark) in a 5D bulk (4D bdry) theory

◇Drag force

The momentum loosing of the string flowing to the horizon

$$F_{drag} = \Pi_\xi = \frac{\partial \mathcal{L}}{\partial \xi'} = -\frac{v\, G_{11}(u_c)}{2\pi\alpha'}$$

Considering a relativistic moving quark

$$F_{drag} = \frac{dp}{dt} = -\eta_D^L p, \qquad p = M\, v\gamma, \qquad \gamma = \left(1 - v^2\right)^{-1/2}$$

friction coefficient is given by

$$\eta_D^L = \frac{G_{11}(u_c)}{2\pi\alpha' M\gamma}$$

Note: At this level there is no fluctuation.

◇Adding fluctuations

The ansatz becomes

$$t=\sigma^0,\quad u=\sigma^1,\quad X^1=v\,t+\xi(u)+\delta X^1(\tau,\sigma),\quad X^i=\delta X^i(\tau,\sigma),\ i=2,3$$

Expanding the NG-action in fluctuations around the classical solution

$$S_2=-\frac{1}{2\pi\alpha'}\int dtdu\sqrt{-g}\,\frac{g^{\mu\nu}}{2}\left[\frac{G_{tt}G_{11}+C^2}{G_{tt}+G_{11}v^2}\partial_\mu\delta X^1\,\partial_\nu\delta X^1\right.$$
$$\left.+\sum_{i=2,3}G_{ii}\partial_\mu\delta X^i\,\partial_\nu\delta X^i\right]$$

no mass term $\Rightarrow$ We can use Membrane paradigm

no mixing term $\Rightarrow$ No off-diagonal components in friction and diffusion coefficients

◇Membrane paradigm

[Iqbal, Liu ('09)]

From the action of generic massless fluctuations,

$$S_{\text{fluc}} = -\frac{1}{2}\int dtdu\sqrt{-g}\,\Lambda(u)\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

one can read off the associated transport coefficient directly from their effective coupling using Green-Kubo fomula,

$$\Lambda(u_h) = -\lim_{\omega\to 0}\left(\frac{\mathrm{Im}\,G_R^{ij}(\omega)}{\omega}\right).$$

On the other hand

$$\kappa^{ij} = -2T_{ws}\lim_{\omega\to 0}\left(\frac{\mathrm{Im}\,G_R^{ij}(\omega)}{\omega}\right)$$

Casalderrey-Solana, Teaney '07

Brownian motion

◇Membrane paradigm

Therefore for Brownian motion

$$S_2 = -\frac{1}{2\pi\alpha'}\int dtdu\sqrt{-g}\,\frac{g^{\mu\nu}}{2}\left[\frac{G_{tt}G_{11}+C^2}{G_{tt}+G_{11}v^2}\partial_\mu\delta X^1\,\partial_\nu\delta X^1 + \sum_{i=2,3} G_{ii}\partial_\mu\delta X^i\,\partial_\nu\delta X^i\right]$$

we find

$$\kappa_T{}^{i=2,3} = \frac{T_{ws}}{\pi\alpha'}\,G_{ii}\Big|_{u=u_c}, \qquad \kappa_L = \frac{T_{ws}}{\pi\alpha'}\,\frac{(G_{tt}\,G_{11})'}{G'_{tt}+v^2\,G'_{11}}\Bigg|_{u=u_c},$$

$$\frac{\kappa_L}{\kappa_T^{i=2,3}} = \frac{G_{11}}{G_{ii}}\left(1-\frac{2\,v^2\,G'_{11}}{G'_{tt}+v^2G'_{11}}\right)\Bigg|_{u=u_c} >_{iso} 1,$$

$$T_{ws} = \frac{1}{4\pi}\left[\left|\frac{G'^2_{tt}-v^4\,G'^2_{11}}{G_{tt}G_{uu}}\right|\right]^{1/2}_{u=u_c}$$

◇Bulk & Boundary agreement

From boundary FT we know

$$\kappa_T^{(FT)} = 2TM\gamma\eta_D^T,$$
$$\kappa_L^{(FT)} = 2TM\gamma^3 \left(\eta_D^L + p\frac{\partial \eta_D^L}{\partial p}\bigg|_{p=p_0} \right).$$

From holography we found

$$\eta_D^L = \frac{G_{11}(u_c)}{2\pi\alpha' M\gamma}, \qquad \eta_D^T = \frac{G_{ii}(u_c)}{2\pi\alpha' M\gamma}$$

$$\kappa_T^{(hol.)} = \frac{T_{ws}}{\pi\alpha'} G_{ii}(u_c), \qquad \kappa_L^{(hol.)} = \frac{T_{ws}}{\pi\alpha'} \frac{(G_{tt}\, G_{11})'}{G'_{tt} + v^2\, G'_{11}}\bigg|_{u=u_c},$$

Chain rule: $\frac{\partial}{\partial p} = \frac{\partial u_c}{\partial p}\frac{\partial}{\partial uc}$ & $v^2 = -G_{tt}(u_c)/G_{11}(u_c)$,

$\Rightarrow \quad \kappa_L^{(hol.)} = \kappa_L^{(FT)}$

◇Einstein equation

In non-relativistic case the Einstein equation is

$$\tau\kappa = 2\,M\,T, \qquad \tau = \frac{1}{\eta_D}$$

In relativistic limit we find the generalized Einstein relation

$$\tau_T\,\kappa_T = 2M\gamma T_{ws}, \qquad \tau_T = \frac{1}{\eta_D^T}$$

where τ is the momentum diffusion time

T_{ws} is the World-Sheet temperature

NOTE: Generalized Einstein relation is defined in terms of a set of physical boundary quantities, and the geometric quantity T_{ws}.

In a sense, T_{ws} is the temperature reads by a quark as it moves through the medium.

◇Relation to shear viscosity?!

In anisotropic background

[Rebhan, Steineder ('11)]

$$\frac{\eta^{(shear)}}{s} = \frac{1}{4\pi}\,\frac{G_{ii}(u_h)}{G_{jj}(u_h)}\ ,$$

Naively

$$\tau_L\,\kappa_T = 2M\gamma\,T_{ws}\frac{G_{ii}(u_c)}{G_{11}(u_c)}, \qquad \tau_L = \frac{1}{\eta_D^L}$$

In non-relativistic limit

$$\tau_L\,\kappa_T = 8\,\pi\,M\,T\,\frac{\eta^{(shear)}}{s}$$

Comments are more than welcome!

[Gubser (06)]

The metric is given by

$$ds^2 = \frac{u^2}{R^2}\left(-f(u)\,dt^2 + d\vec{x}^2\right) + \frac{R^2}{u^2 f(u)}du^2, \qquad f(u) = 1 - \frac{u_h^4}{u^4}$$

It is easy to show

$$T_{ws} = T\,\left(1 - v^2\right)^{1/4}, \qquad \eta_D = \frac{\pi\sqrt{\lambda}T^2}{2M}$$

and

$$K_L = \frac{K_T}{1 - v^2} = \frac{\sqrt{\lambda}\pi T^3}{(1 - v^2)^{5/4}}, \qquad \sqrt{\lambda} = \frac{R^2}{2\pi\alpha'}$$

So

$$\kappa_L > \kappa_T$$

Anisotropic asymptotically AdS$_5$ black brane

[Mateos, Trancanelly (11)]

Gauge Theory

Gravity

$$S = S_{\mathcal{N}=4} + \frac{1}{8\pi^2}\int \theta(z) \mathrm{Tr}\,(F \wedge F)$$

D-branes	t	x	y	z	u	S^5
N$_c$ D3	×	×	×	×		
n$_7$ D7	×	×	×			×

$\theta(z) = az$

D7s are dissolved (full-back reaction), $n_7 = \frac{dN_{D7}}{dz}$

D7s don't extend in u $\Rightarrow$ NO new degrees of freedom

Complexified coupling

$$\tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g_{YM}^2} = \chi + i\,e^{-\phi}$$

Anisotropic asymptotically AdS$_5$ black brane

[Mateos, Trancanelly (11)]

Gauge Theory | **Gravity**

$$S = S_{\mathcal{N}=4} + \frac{1}{8\pi^2}\int \theta(z)\,\mathrm{Tr}\,(F \wedge F)$$

D-branes	t	x	y	z	u	S^5
N_c D3	$\times$	$\times$	$\times$	$\times$		
n_7 D7	$\times$	$\times$	$\times$			$\times$

Axion is sourced by D7s $\Rightarrow$ 5D axion-dilaton-gravity

$$S = \frac{1}{16\pi G}\int \sqrt{-G}\left[R - 2\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\right] + \text{bdry. term}$$

◇ Anisotropic Anstaz

$$ds^2 = \frac{e^{-\phi/2}}{u^2}\left(-\mathcal{F}\,\mathcal{B}\,dt^2 + dx^2 + dy^2 + \mathcal{H}dz^2 + \frac{du^2}{\mathcal{F}}\right),$$

$$\phi = \phi(u)\,, \qquad \chi = a\,z\,, \qquad a := \frac{\lambda\, n_7}{4\pi N_c}\,.$$

eoms $\Rightarrow$ $\mathcal{H} = e^{-\phi}$

eoms $+$ $\phi(0) = 0$ $\Rightarrow$ $\mathcal{F}(0) = \mathcal{H}(0) = \mathcal{B}(0) = 1$

RG-flow

AdS at UV $\longrightarrow$ Lifshitz-like at IR

with

$$T = \frac{|\mathcal{F}'_H|\sqrt{\mathcal{B}_H}}{4\pi}$$

Anisotropic asymptotically AdS_5 black brane

◇ Some properties

- Static solution
- Anisotropic horizon
- Regular on & out of the horizon
- $\partial^\mu T_{\mu\nu} = 0$
- $< T^\mu_\mu > \sim N_c^2 \, a^4$
- Homogeneous & in-homogeneous phases (instabilities)
- Naked curvature singularity at zero temperature
  [Azeyanagi, Li, Takayanagi('09)]
- No new degrees of freedom to SYM (usual AdS/CFT is applicable)
- Solution is known analytically for small $\frac{a}{T}$ $\left(\mathcal{O}\left(a^{2n}\right)\right)$
- In large anisotropy $s \sim a^{1/3} \, T^{8/3}$
- The KSS bound is violated $\eta/s < 1$.

[Rebhan, Steineder('11)]

Singular background

[Janik, Witaszczyk ('08)]

◇ Metric in Fefferman-Graham expansion is given by

$$ds^2 = \frac{1}{u^2}\,\left(-a(u)\,dt^2 + b(u)\,\left(dx_1^2 + dx_2^2\right) + c(u)\,dx_3^2 + du^2\right)\;,$$

where u is the radial coordinate with the boundary at $u = 0$ and metric functions are defined by boundary conditions as following,

$$\begin{aligned} a(u) &= \left(1 + A^2\,u^4\right)^{\frac{1}{2}-\frac{1}{4}\sqrt{36-2B^2}}\,\left(1 - A^2\,u^4\right)^{\frac{1}{2}+\frac{1}{4}\sqrt{36-2B^2}}\,, \\ b(u) &= \left(1 + A^2\,u^4\right)^{\frac{1}{2}+\frac{B}{6}+\frac{1}{12}\sqrt{36-2B^2}}\,\left(1 - A^2\,u^4\right)^{\frac{1}{2}-\frac{B}{6}-\frac{1}{12}\sqrt{36-2B^2}}\,, \\ c(u) &= \left(1 + A^2\,u^4\right)^{\frac{1}{2}-\frac{B}{3}+\frac{1}{12}\sqrt{36-2B^2}}\,\left(1 - A^2\,u^4\right)^{\frac{1}{2}+\frac{B}{3}-\frac{1}{12}\sqrt{36-2B^2}}\,. \end{aligned}$$

NOTE: The metric is singular at $u = A^{-1/2}$, except for $B = 0$!

Singular background

◇ Energy and pressures for boundary theory

Using Fefferman-Graham expansion

$$\varepsilon = \frac{A^2}{2}\sqrt{36 - 2B^2}\ ,$$

$$P^{\parallel} = \frac{A^2}{6}\sqrt{36 - 2B^2} - \frac{2}{3}A^2 B, \qquad P^{\perp} = \frac{A^2}{6}\sqrt{36 - 2B^2} + \frac{1}{3}A^2 B.$$

associated "temperature"

$$T^4 = \frac{8\,\varepsilon}{3\,\pi^2\,N_c^2} = \frac{4\,A^2\,\sqrt{36 - 2B^2}}{3\,\pi^2\,N_c^2}\ ,$$

Oblate	Prolate
$B = \sqrt{2}$	$B = -\sqrt{6}$
$P^{\parallel} = 0$	$P^{\perp} = 0$
$g^{\parallel} < g^{\perp}$	$g^{\parallel} > g^{\perp}$

Brownian motion in anisotropic plasmas

Reminder: for a moving quark in X^1 direction

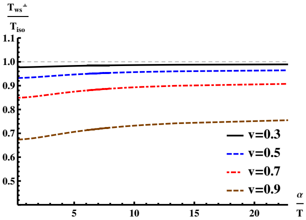
$$T_{ws} = \frac{1}{4\pi}\left[\left|\frac{G_{tt}'^2 - v^4\, G_{11}'^2}{G_{tt}\, G_{uu}}\right|\right]^{1/2}_{u=u_c}$$

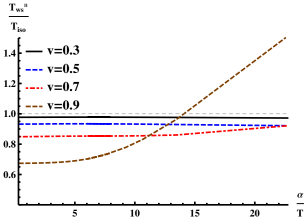
$$\kappa_T{}^{i=2,3} = \frac{T_{ws}}{\pi\alpha'}\, G_{ii}\Big|_{u=u_c}, \qquad \kappa_L = \frac{T_{ws}}{\pi\alpha'}\, \frac{(G_{tt}\, G_{11})'}{G_{tt}' + v^2\, G_{11}'}\Bigg|_{u=u_c},$$

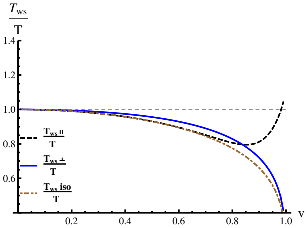
$$\frac{\kappa_L}{\kappa_T^{i=2,3}} = \frac{G_{11}}{G_{ii}}\left(1 - \frac{2\, v^2\, G_{11}'}{G_{tt}' + v^2 G_{11}'}\right)\Bigg|_{u=u_c},$$

According to anisotropy direction there are
For a moving quark in anisotropy direction: $\kappa_L^{\parallel}, \kappa_T^{\parallel}$
For a moving quark in a transverse direction $\kappa_L^{\perp}, \kappa_T^{\perp,(\perp)}, \kappa_T^{\perp,(\parallel)}$

◇ World-sheet temperatures

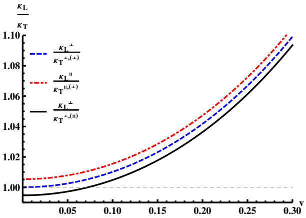




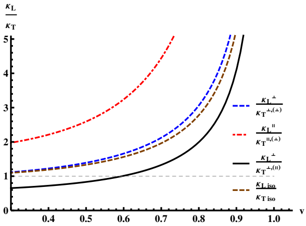


$a/T \sim 10$

◇ Langevin diffusion coefficients



$a/T \sim 0.5$



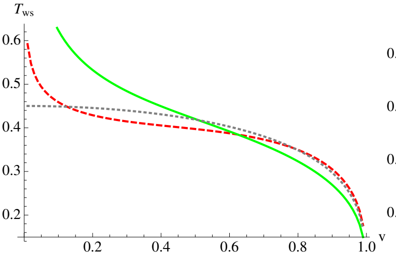
$a/T \sim 10$

Violation from isotropic inequality ($\kappa_L > \kappa_T$) is only in small enough velocity of $\frac{\kappa_L^{\perp}}{\kappa_T^{\perp,(\|)}}$.

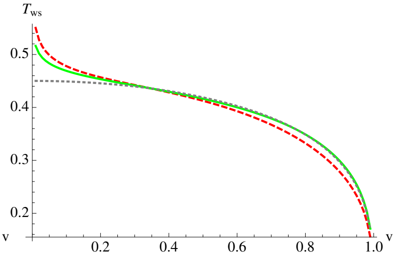
Brownian motion in anisotropic bottom-up plasma

◇ World-sheet temperature

There is no hawking temperature.



Prolate



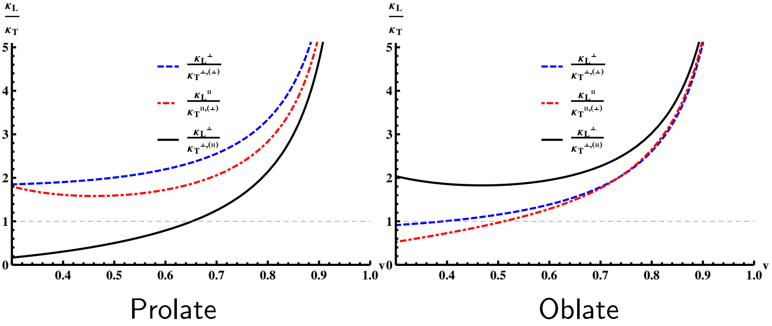
Oblate

red dash : moving along anisotropy

green lin: moving transverse to anisotropy

gray dot: isotropic ($B = 0$)

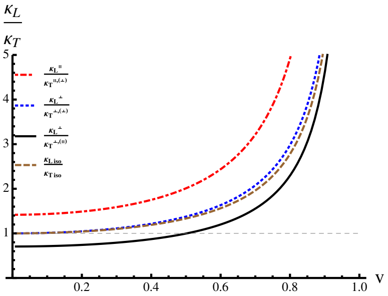
◇ Langevin diffusion coefficients



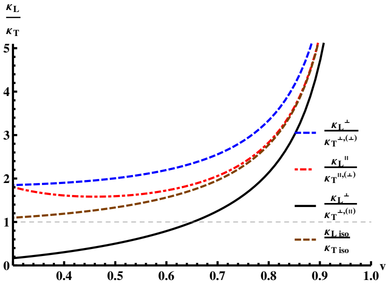
Different branches may violate the isotropic inequality!

◇ Common Results Between the two Models

A way to relate the two models is to fix the $\Delta := \frac{P^{\perp}}{P^{\parallel}} - 1$.



Top-down ($a/T \simeq 6.43$)



Bottom-up ($B \sim -\sqrt{6}$)

Both of them are prolate in geometry and momentum ($\Delta = -1$)!

Summary & Conclusion

- We compute the T_{ws} for a moving heavy quark in a general static background.
- In anisotropic media, the effective temperature might be smaller or larger than the medium's temperature: "refrigerator" or "heater".
- We calculate the Langevin diffusion coefficients $\kappa_{L,T}$.
- We find a perfect agreement between holographic and boundary FT broadening parameters.
- Einstein relation is generalized and a relation to Shear viscosity is proposed!
- In the known cases we found the agreement with the previous results e.g. IHQCD.
- Quarks moving in different directions feel different effective temperatures and different diffusion coefficients.
- Violation from isotropic inequality ($\kappa_L > \kappa_T$) might happen in anisotropic media.

Thank you for your attention!