On 4D Scale Invariant Gravity

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Based on

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See also

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Useful references

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S. Deser, H. Liu, H. Lu, C. N. Pope, T. C. Sisman and B. Tekin, "Critical Points of D-Dimensional Extended Gravities," Phys. Rev. D **83**, 061502 (2011) [arXiv:1101.4009 [hep-th]].

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It is interesting to study R^2 gravity in four dimensions

- R^2 gravity would naturally appear from string theory compactifications.

- Inflationary models based on R^2 gravity seem interesting!

- It might be possible to construct a ghost free gravity.

As usual we start with more extra symmetries which could make the model simpler. Then we could break the extra symmetry properly.

Scale Invariant Gravity

The most general action of a scale invariant four dimensional gravity can be written as follows

$$I_{SC} = -\frac{\kappa}{32\pi}\int d^4x\,\sqrt{-g}\,\left(\sigma_0 C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} + R^2 + \sigma_1 \mathrm{GB}_4\right).$$

where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor and σ_0, σ_1 and κ are dimensionless constant.

We will try to fix the corresponding action as much as we could using certain natural assumptions based on the holographic renormalization.

To proceed we will consider an asymptotically AdS geometry which could solve the equations of motion as follows

$$ds^2 = \frac{L^2}{r^2}\left(dr^2 + g_{ij}(x)dx^i dx^j\right), \qquad \text{with } \; g_{ij}(x) = \delta_{ij} + h_{ij}(x)r^2 + \mathcal{O}(r^3).$$

For this solution the on shell action reads

$$I_{SC} = -\frac{\kappa}{32\pi}\int d^4x \left[\sigma_0\mathcal{O}(1) + \left(\frac{144}{r^4} - \frac{6a_2}{r^2} + \mathcal{O}(1)\right) + +\sigma_1\left(\frac{24}{r^4} - \frac{a_2}{r^2} + \mathcal{O}(1)\right)\right].$$

a_2 is given in terms of the parameters appearing in the asymptotic expansion of the metric. In order to get a finite action one should set $\sigma_1 = -6$. As a result we will consider the action of 4D scale invariant gravity as follows

$$I_{SC} = -\frac{\kappa}{32\pi}\int d^4x\,\sqrt{-g}\,\left(\sigma_0 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + R^2 - 6\mathrm{GB}_4\right),$$

which is guaranteed to get finite on shell action for asymptotically AdS geometries.

The most general action of scale invariant (pure) gravity in four dimensions is a one family parameter action.

The aim is to study the model on a generic point of the moduli space of the parameter.

- We recover conformal gravity at point $\sigma_0 \to \infty$ which is ghost free theory for Einstein solutions.

- At $\sigma_0 = 0$ one gets pure R^2 gravity which has recently been studied, extensively.

The equations of motion are

$$\left(\nabla^\sigma \nabla^\rho - \frac{1}{2} R^{\sigma\rho}\right) C_{\mu\sigma\nu\rho} = \frac{1}{2\sigma_0} \left(R R_{\mu\nu} - g_{\mu\nu} \frac{R^2}{4} - \nabla_\mu \nabla_\nu R + g_{\mu\nu} \Box R \right),$$

where $\Box = \nabla^\mu \nabla_\mu$. These equations of motion admit several black hole solutions. Indeed, restricting to Einstein solutions,

$$R_{\mu\nu} = \frac{3\lambda}{L^2} g_{\mu\nu}, \quad \lambda = \pm 1, 0$$

the above equations of motion are solved by the following black hole solutions

$$ds^2 = \frac{L^2}{r^2} \left(-F(r)\, dt^2 + \frac{dr^2}{F(r)} + d\Sigma^2_{2,k} \right), \qquad F(r) = \lambda + k r^2 + c_3 r^3,$$

where L being the radius of curvature and $k = 1, -1, 0$ corresponds to $\Sigma_{2,k} = S^2, H_2, R^2$, respectively.

These are Einstein solutions in which both sides of the equations of motion vanish identically. It is then natural to look to a solution of whole equation on which the two sides of the equation do not vanish separately. To find a new solution one may start from the following AdS wave anstaz

$$ds^2 = \frac{L^2}{r^2}\left(dr^2 + dy^2 - 2dx_- dx_+ + k(x_+, r)\, dx_+^2\right).$$

For generic σ_0 one gets the following general solution

$$k(x_+, r) = c_0(x_+) + c_3(x_+) r^3 + b_1(x_+) r^{\frac{3}{2} - \frac{1}{2}\sqrt{\frac{\sigma_0 + 48}{\sigma_0}}} + b_2(x_+) r^{\frac{3}{2} + \frac{1}{2}\sqrt{\frac{\sigma_0 + 48}{\sigma_0}}}.$$

In particular it admits a logarithmic solution for $\sigma_0 = 6$

$$k(x_+, r) = c_0(x_+) + b_0(x_+) \log r + \Big(c_3(x_+) + b_3(x_+) \log r\Big) r^3.$$

It is illustrative to evaluate the on shell action (free energy) for the above black hole solutions which in turns can be used to compute the corresponding entropy of the solutions.

$$I_{SG} = -\frac{\kappa}{32\pi}(V_2\beta)\int_0^{r_H} dr\ 12c_3^2 r^2(\sigma_0 - 6) = -\frac{\kappa V_2}{2}\,(\sigma_0 - 6)\,\frac{(\lambda + kr_H^2)^2}{r_H^2(3 + kr_H^2)}$$

where β is the period of the Euclidean time which is given in terms of the Hawking temperature $\beta = T_H^{-1}$, and V_2 is the volume of the two dimensional internal space $\Sigma_{2,k}$.

Note that the on shell action vanishes at $\sigma_0 = 6$ which could result to zero entropy for the corresponding black hole solutions.

Therefore one would expect that the model should admits a new feature at this *critical point*. It is worth noting that to find the significant of the critical point $\sigma_0 = 6$ from free energy computation (or entropy) its was crucial to take into account the contribution of Gauss-Bonnet term.

To understand this point better let us consider the metric perturbation $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ where the vacuum background metric $\bar{g}_{\mu\nu}$ satisfies the following relations

$$\bar{R}_{\mu\rho\nu\sigma} = \frac{\Lambda}{3}(\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\mu\rho}), \qquad \bar{R}_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}, \qquad \bar{R} = 4\Lambda.$$

For this vacuum the linearized equations of motion are

$$\frac{4\Lambda}{3}(6-\sigma_0)\,\mathcal{G}^{(1)}_{\mu\nu} + \frac{2}{3}(\sigma_0+3)\Big(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \Lambda\bar{g}_{\mu\nu}\Big)R^{(1)}$$
$$+2\sigma_0\Big((\bar{\Box} - \frac{2\Lambda}{3})\mathcal{G}^{(1)}_{\mu\nu} - \frac{2\Lambda}{3}\bar{g}_{\mu\nu}R^{(1)}\Big) = 0,$$

where

$$\mathcal{G}^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(1)} - \Lambda h_{\mu\nu}.$$

with $R^{(1)}_{\mu\nu}$ and $R^{(1)}$ are linearized Ricci tensor and Ricci scaler, respectively.

The linearized equations of motion may be studied in different gauges. In particular in the traceless-transverse gauge where $\bar{\nabla}^{\mu} h_{\mu\nu} = h = 0$, the above equation reads

$$\left[\frac{2\Lambda}{3}(6-\sigma_0) + \sigma_0\left(\bar{\Box} - \frac{2\Lambda}{3}\right)\right]\left(\bar{\Box} - \frac{2\Lambda}{3}\right) h_{\mu\nu} = 0.$$

Therefore the model has two modes: massive and massless given by

$$\left(\bar{\Box} - \frac{2\Lambda}{3}\right) h^{(0)}_{\mu\nu} = 0, \qquad \left[\frac{2\Lambda}{3}(6-\sigma_0) + \sigma_0\left(\bar{\Box} - \frac{2\Lambda}{3}\right)\right] h^{(m)}_{\mu\nu} = 0,$$

whose energies are found

$$\begin{aligned} E^{(0)} &= \frac{2}{3}(6-\sigma_0)\int d^4\sqrt{-\bar{g}}\; h^{(0)}_{\mu\nu}\bar{\bar{\nabla}}^0 h^{(0)\mu\nu}, \\ E^{(m)} &= -\frac{2}{3}(6-\sigma_0)\int d^4\sqrt{-\bar{g}}\; h^{(m)}_{\mu\nu}\bar{\bar{\nabla}}^0 h^{(m)\mu\nu} \end{aligned}$$

that have opposite signs, though at the point $\sigma_0 = 6$ they vanish.

Indeed at this point the equations on motion degenerates leading to the logarithmic solution.

$\sigma_0 = 0$; R^2-gravity revisited

In this case the equations of motion are

$$\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right)R + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)R = 0.$$

It is then clear that all Einstein solutions presented in the previous section are still solutions of the above equations.

It seems that the model looks a trivial subclass of the general model studied before. though this is not the case.

Actually the model admits a large class of solutions which are not solutions of the general scale invariant gravity. More precisely there is a new class of solutions for which $R = 0$ while $R_{\mu\nu} \neq 0$.

For example one has following new solutions which are Ricci scalar flat

$$ds^2 = \frac{L^2}{r^2}\left(-F(r)\,dt^2 + \frac{dr^2}{F(r)} + d\Omega^2_{2,k}\right), \qquad F(r) = kr^2 + c_3 r^3 + c_4 r^4.$$

where c_3 and c_4 are constant of the integration. It is easy to see that for this solution $R = 0$ while as long as $c_4 \neq 0$ the Ricci tensor is non-zero. More precisely one has

$$R_{\mu\nu} = -\frac{c_4 r^4}{2L^2}\,(\eta^\rho_\mu g_{\rho\nu} + \eta^\rho_\nu g_{\rho\mu}), \qquad \text{with } \eta^\nu_\mu = \text{diag}(-1,-1,1,1).$$

Actually these solutions were missed for several years! Taking into account these solutions R^2 gravity is not equivalent to Einstein gravity coupled to a scalar field.

It is also interesting to study the above solution for particular values of parameters.

For $k = 1, c_4 = c^2$ and $c_3 = -2c$, setting $r = \frac{1}{\xi}$ one gets

$$ds^2 = L^2 \left[-\left(1 - \frac{c}{\xi}\right)^2 dt^2 + \frac{d\xi^2}{\left(1 - \frac{c}{\xi}\right)^2} + \xi^2 d\Omega_2^2 \right],$$

which is an extremal black hole whose near horizon geometry is $\mathrm{AdS}_2 \times S^2$.

For $k = 0$ and $c_4 = 1$, setting $r = \frac{1}{\rho}$, one finds

$$ds^2 = L^2 \rho^2 \left(-(1 + c_3 \rho) \frac{dt^2}{\rho^4} + \frac{d\rho^2}{1 + c_3 \rho} + dx_1^2 + dx_2^2 \right)$$

which is recognized as a metric with hyperscaling violating factor whose hypescaling violation and anisotropic exponents are $\theta = 4, z = 3$.

Let us recall that in general a $d+2$ dimensional hyperscaling violating geometry may be written as follows

$$ds^2 = \frac{L^2}{r^2} r^{2\frac{\theta}{d}} \left(-\frac{f(r)}{r^{2(z-1)}} dt^2 + \frac{dr^2}{f(r)} + d\vec{x}_d^2 \right), \qquad \text{with } f(r) = 1 - m\, r^{d-\theta+z}.$$

$$S \sim T^{\frac{d-\theta}{z}}$$

It is also interesting to study small perturbations above a Ricci scalar flat solution where one has $R = 0$ while $R_{\mu\nu} \neq 0$. To proceed let us consider the perturbation $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ with $\bar{g}_{\mu\nu}$ being the background metric

$$\bar{R} = 0, \qquad \bar{R}_{\mu\nu} = -\frac{c_4 r^4}{2L^2}(\eta^{\alpha}_{\mu}\bar{g}_{\alpha\nu} + \eta^{\alpha}_{\nu}\bar{g}_{\alpha\mu}).$$

Using this perturbation the equations of motion at linear level read

$$\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu} + \bar{R}_{\mu\nu}\right)R^{(1)} = 0.$$

where

$$R^{(1)} = -\bar{\Box}h + \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}h^{\mu\nu} - \bar{R}_{\mu\nu}h^{\mu\nu}.$$

So that

$$\left(\bar{g}_{\mu\nu}\bar{\nabla}^2 - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu} + \bar{R}_{\mu\nu}\right)\left(g_{\alpha\beta}\bar{\Box} - \bar{\nabla}_{\alpha}\bar{\nabla}_{\beta} + \bar{R}_{\alpha\beta}\right)h^{\alpha\beta} = 0.$$

One may decompose the metric fluctuations as follows

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \bar{\nabla}_\mu V_\nu + \bar{\nabla}_\nu V_\mu + \left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{1}{4}\bar{g}_{\mu\nu}\bar{\Box}\right) V + \frac{1}{4}\bar{g}_{\mu\nu}\phi,$$

where $\hat{h}_{\mu\nu}$ is a traceless transverse tensor: $\bar{g}^{\mu\nu}\hat{h}_{\mu\nu} = 0$, $\bar{\nabla}^\mu \hat{h}_{\mu\nu} = 0$, and V_μ is a divergenceless vector: $\bar{\nabla}^\mu V_\mu = 0$. V and $\phi = h$ are two scalars.

On the other hand gauge transformations are the infinitesimal diffeomorphisms $x^\mu \to x^\mu + \xi^\mu(x)$ under which the metric transforms as follows

$$h_{\mu\nu} \to h_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu.$$

It is then useful to decompose the gauge transformation parameter ξ^μ into transverse and longitudinal parts $\xi^\mu = \hat{\xi}^\mu + \bar{\nabla}^\mu \xi$ so that $\bar{\nabla}^\mu \hat{\xi}_\mu = 0$.

under the gauge transformation transform one has

$$\hat{h}_{\mu\nu} \to \hat{h}_{\mu\nu}, \qquad V_\mu \to V_\mu + \hat{\xi}_\mu, \qquad V \to V + 2\xi, \qquad \phi \to \phi + 2\bar{\Box}\xi.$$

The gauge invariant fields may be defined as follows

$$\gamma_{\mu\nu} = \hat{h}_{\mu\nu}, \qquad \Phi = \phi - \bar{\Box}V, \qquad \Psi = \bar{\nabla}^\mu V_\mu.$$

It is then important to rewire the equations of motion for the gauge invariant fields. To do so we note that

$$\bar{\nabla}^{\mu}\bar{\nabla}^{\nu}h_{\mu\nu} = 2\bar{R}^{\mu\nu}\bar{\nabla}_{\mu}V_{\nu} + \left(\bar{R}^{\mu\nu}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu} + \frac{3}{4}\bar{\Box}^{2}\right)V + \frac{1}{4}\bar{\Box}\phi,$$

$$\bar{R}^{\mu\nu}h_{\mu\nu} = \bar{R}^{\mu\nu}\hat{h}_{\mu\nu} + 2\bar{R}^{\mu\nu}\bar{\nabla}_{\mu}V_{\nu} + \bar{R}^{\mu\nu}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}V$$

Thus the Ricci scalar at first order reads

$$R^{(1)} = -\frac{3}{4}\bar{\Box}\Phi - \bar{R}_{\alpha\beta}\gamma^{\alpha\beta} = -\frac{3}{4}\bar{\Box}\Phi + \frac{c_4 r^4}{L^2}\eta^{\mu}_{\nu}\gamma^{\nu}_{\mu}.$$

Therefore the equations of motion can be recast to the following form

$$\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu} + \bar{R}_{\mu\nu}\right)\left(\bar{\Box}\Phi - \frac{4c_4 r^4}{3L^2}\eta^{\alpha}_{\beta}\gamma^{\beta}_{\alpha}\right) = 0,$$

whose trace is

$$\bar{\Box}\left(\bar{\Box}\Phi - \frac{4c_4 r^4}{3L^2}\eta^{\alpha}_{\beta}\gamma^{\beta}_{\alpha}\right) = 0,$$

If one assumes that the metric fluctuations above a Ricci scalar flat geometry are themselves Ricci scalar flat, then

$$\bar{\Box}\Phi - \frac{4c_4 r^4}{3L^2}\eta^{\alpha}_{\beta}\gamma^{\beta}_{\alpha} = 0$$

On the other hand for $R^{(1)} \neq 0$ then the new combination

$$\psi = \bar{\Box}\Phi - \frac{4c_4 r^4}{3L^2}\eta^{\alpha}_{\beta}\gamma^{\beta}_{\alpha}$$

defines a non-zero field satisfying the following scalar equation

$$\bar{\Box}\psi = 0,$$

There is no gravitons!

What we learn from these results

Scale invariant gravity in four dimensions has one parameter family action.

In different point of moduli space of the parameter the model exhibits different behaviors. In particular there are three critical points in the moduli space of parameter given by $\sigma_0 = 0, 6$ and $\sigma_0 \to \infty$.

At $\sigma_0 \to \infty$ one recovers the four dimensional conformal gravity.

At $\sigma_0 = 0$ it reduces to pure R^2 gravity.

At the critical point $\sigma_0 = 6$ one indeed arrives at a Log-Gravity where the natural vacuum is a logarithmic solution.

Restricted to Einstein solutions in a generic point of the moduli space above an AdS vacuum the model has a ghost excitation.

Restricted to Einstein solutions the conformal gravity obtained at $\sigma_0 = \infty$ is ghost free.

Therefore imposing to have a ghost free model, one is forced to sit at $\sigma_0 = \infty$ where the full conformal symmetry is restored.

Note that we start with a scale invariant gravity.

Thus scale invariant together with ghost free condition bring us to a conformal gravity!

It is reminiscent of the fact that unitary theory with scale symmetry is indeed a conformal theory.