A Wavy Way Leading to the Kerr Metric and Its Quantum Singularity Analysis

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8 October 2015, IPM-Tehran

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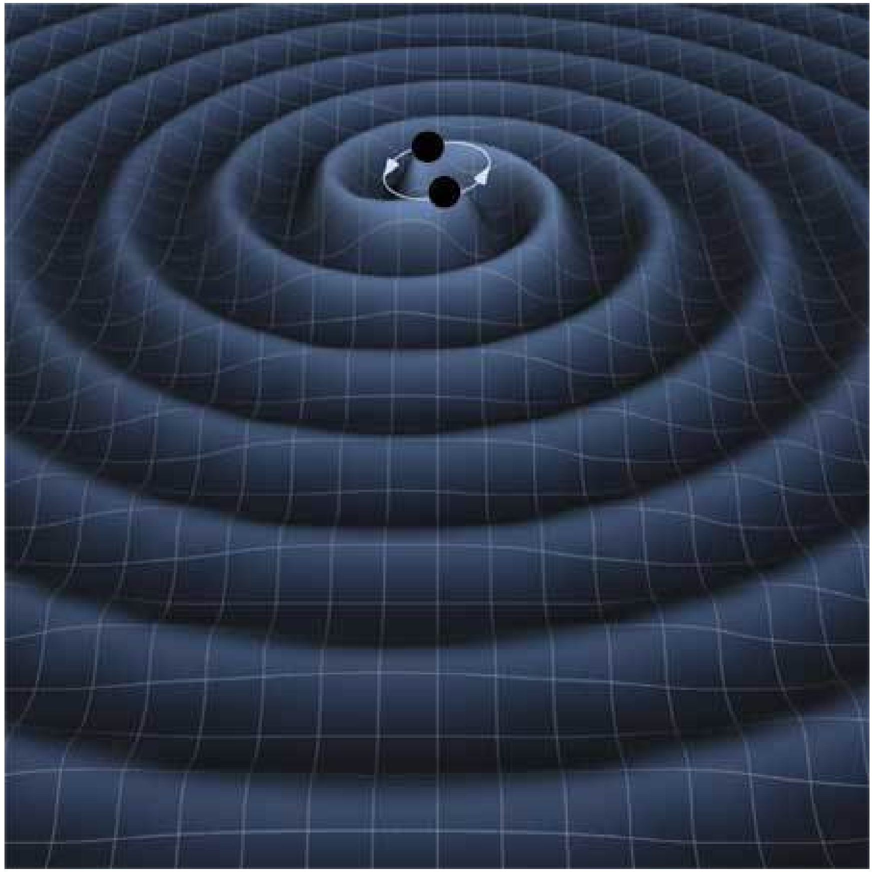
2 Introduction

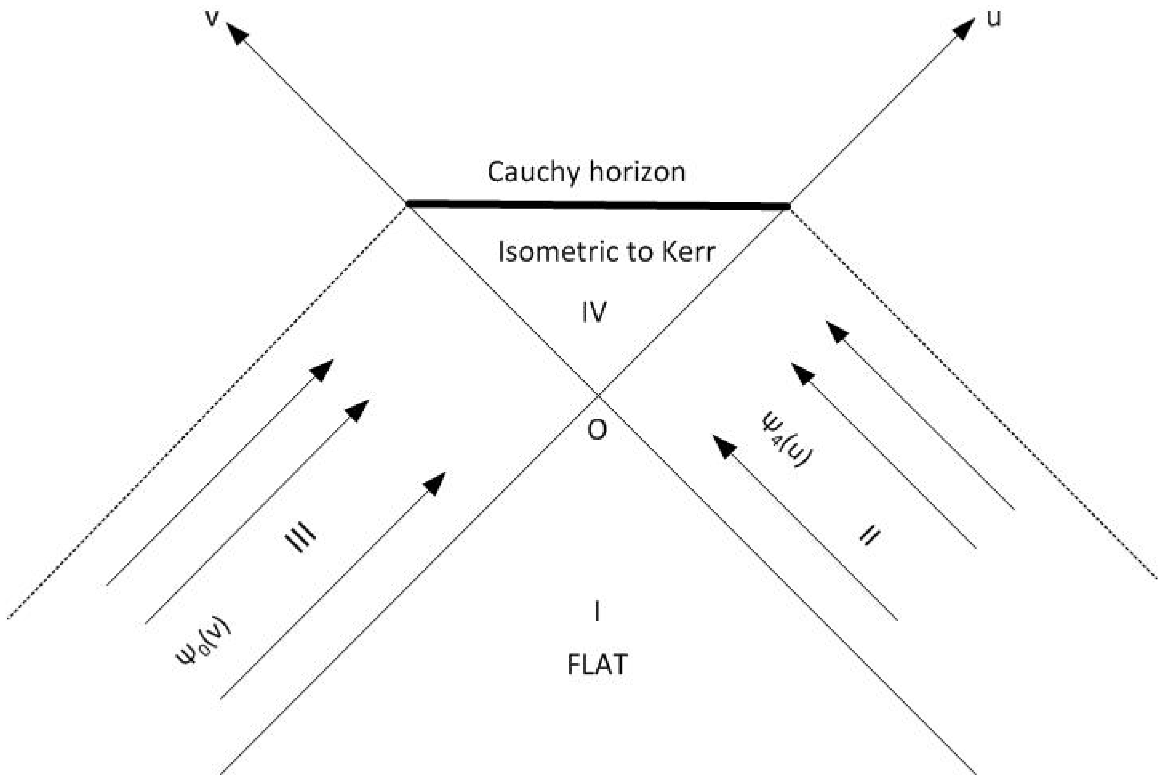
- Gravitational waves, black holes and singularities are the predictions of the Einstein's Theory of Relativity, as a consequence of the solution to the Einstein's equations.

- These three seemingly unrelated topics of General Relativity are contained in the nonlinear interactions of gravitational waves. Namely, *Colliding Gravitational Plane Waves*. (J. B. Griffiths, *Colliding plane waves in general relativity*, Oxford University Press (1991))

- Gravitational waves are described as the ripples in the fabric of spacetime which propagates at the speed of light.



- The formulation of the problem of Colliding Gravitational Plane Waves requires to devise the whole spacetime into four distinct regions.



v
u
Cauchy horizon
Isometric to Kerr
IV
O
III
II
$\Psi_0(v)$
$\Psi_4(u)$
I
FLAT

- Singularities: Are described as "end points" or incomplete geodesics for timelike or null trajectories followed by classical particles.

- According to Ellis and Schmidt (*Gen. Rel. Grav.* **8**, 915, 1977): Classical curvature singularities are grouped as **scalar, nonscalar** and **quasiregular**.

- Singularity theorems are silent about the *"nature"* of the singularity.

  i Spacelike character

  ii Timelike character

  iii Null character

- Naked singularity: Visible to asymptotic observers $\longrightarrow$ Forms a threat to **cosmic censorship hypothesis**.

3 Brief Review of Chandrasekhar and Xanthopoulos Solution

(Ref: *Proc. Roy. Soc. A*, **408**, 175-208,1986)

The adopted line element for the description of the CGW is the Szekeres line element given by,

$$ds^2 = 2e^{-M}dudv - e^{-U}\left\{\left(e^{V}dx^2 + e^{-V}dy^2\right)\cosh W - 2\sinh W dxdy\right\}, \quad (1)$$

in which $M = M(u,v), U = U(u,v), V = V(u,v)$ and $W = W(u,v)$ are the metric functions to be found, all depends on the null coordinates u and v in the region of interaction.

The vacuum Einstein equations governing the solution to the metric functions are derived by varying the following Lagrangian,

$$L = e^{-U}\left\{M_u U_v + M_v U_u + U_u U_v - V_u V_v \cosh^2 W - W_v W_u\right\}, \qquad (2)$$

The vacuum Einstein's equations are:

$$U_{uv} = U_u U_v, \tag{3}$$

$$2U_{vv} = U_v^2 + W_v^2 + V_v^2 \cosh^2 W - 2U_v M_v, \tag{4}$$

$$2U_{uu} = U_u^2 + W_u^2 + V_u^2 \cosh^2 W - 2U_u M_u, \tag{5}$$

$$2V_{uv} = U_u V_v + U_v V_u - 2\left(V_u W_v + V_v W_u\right) \tanh W, \tag{6}$$

$$2M_{uv} = -U_u V_v + W_v W_u + V_u V_v \cosh^2 W, \tag{7}$$

$$2W_{uv} = U_u W_v + U_v W_u + 2V_u V_v \sinh W \cosh W. \tag{8}$$

The set of the above field equations can be solved by employing the Ernst formalism (Ref:*Phys Rev.*, **167**, 1175-1178, (1968)). In doing this, the following complex valued function is defined,

$$Z = \chi + iq_2, \tag{9}$$

where

$$\chi = \frac{e^{-V}}{\cosh W}, \qquad \text{and} \qquad q_2 = e^{-V} \tanh W, \tag{10}$$

such that the line element (1) becomes

$$ds^2 = 2e^{-M} du dv - e^{-U} \left[\chi dy^2 + \frac{1}{\chi} (dx - q_2 dy)^2 \right]. \tag{11}$$

By defining a new set of coordinates,

$$\eta = u\sqrt{1 - v^2} + v\sqrt{1 - u^2}, \qquad \text{and} \qquad \mu = u\sqrt{1 - v^2} - v\sqrt{1 - u^2}, \tag{12}$$

the metric that describes the collision of gravitational waves in the region of interaction is transformed to the following form,

$$ds^2 = e^{\nu+\mu_3}\sqrt{\Delta}\left[\frac{(d\eta)^2}{\Delta} - \frac{(d\mu)^2}{\delta}\right] - \sqrt{\Delta\delta}\left[\chi\,(dy)^2 + \frac{1}{\chi}\,(dx - q_2 dy)^2\right], \quad (13)$$

where η defines the time from the instant of the collision, μ defines the distance in the normal direction to the spacelike $(x, y) - planes$ with,

$$\Delta = 1 - \eta^2, \qquad \text{and} \qquad \delta = 1 - \mu^2, \quad (14)$$

Consequently, the metric that describes the collision of gravitational waves is given by

$$ds^2 = X\left[\frac{(d\eta)^2}{\Delta} - \frac{(d\mu)^2}{\delta}\right] - \Delta\delta\frac{X}{Y}dy^2 - \frac{Y}{X}\,(dx - q_2 dy)^2\,, \quad (15)$$

where

$$X=(1-p\eta)^2+q^2\mu^2, \qquad Y=1-\left|\left(E^\dagger\right)\right|^2=1-p^2\eta^2-q^2\mu^2=p^2\Delta+q^2\delta, \tag{16}$$

and

$$q_2=\frac{2q}{p\,(1+p)}-\frac{2q\delta\,(1-p\eta)}{pY}, \tag{17}$$

The Petrov classification of the metric (15), as is shown by CX is type - D. Calculations for the Weyl scalar with a proper tetrads reveals the only nonvanishing scalar as

$$\Psi_2=\frac{1}{2\,(1-p\eta-iq\mu)^3}. \tag{18}$$

The Weyl scalar Ψ_2, in the terminology of the CGW is interpreted as the Coulomb component and arises as aresult of the non-linear interaction. The unboudedness of

Ψ_2 indicates the existence of the curvature singularity. The focussing hypersurface corresponds to $\eta = 1$, for the metric (15). And hence, the behaviour of Ψ_2 is finite which indicates Killing - Cauchy horizon instead of a curvature singularity.

3.1 Extension of the Interaction Region into Incoming Regions

In order to find the wave profiles that participate in the collision, the metric obtained in the interaction region should be extended to the incoming regions. The approaching wave in one of the plane symmetric Region II $(u \geq 0, v < 0)$ is obtained by dropping the v in the metric (15). This is achieved by the substitution $\eta = \mu = \sin(u\theta(u))$, so that the metric functions take the form

$$X(u) = 1 - 2p \sin u + \sin^2 u, \qquad Y(u) = \Delta = \delta = \cos^2 u, \tag{19}$$

$$q_2(u) = \frac{2q}{(1+p)} \left[(1+p)\sin u - 1\right]. \tag{20}$$

in which as described above u is implied with the step function. Hence, the metric in Region II in terms of the null coordinate u can be expressed as

$$ds^2 = \frac{2X(u)}{\sqrt{1-u^2}} du dv - \left(1-u^2\right)\left[X(u)dy^2 + \frac{1}{X(u)}\left(d\widetilde{x} - 2q\sin u dy\right)^2\right], \tag{21}$$

where

$$\widetilde{x} = x + \frac{2q}{(1+p)} y. \tag{22}$$

The plane symmetric metric (21) has a single curvature tensor component which describes the profile of the incoming gravitational wave given by

$$\Psi_4 = -\left(p - iq\right)\delta\left(u\right) - \frac{3\left(X - 2iq\sin u\right)}{X^4\sqrt{X^2 + 4q^2\sin^2 u}}\frac{\left(1 - p\sin u - iq\sin u\right)^3}{\left(p + iq\right)^2}\theta(u), \quad (23)$$

in which $\delta\left(u\right)$ stands for the Dirac delta function and $X = X\left(u\right)$ is given (19). Consequently, the incoming wave is a composition of an impulsive and shock gravitational waves. Similar incoming wave profile $\Psi_0\left(v\right)$ from the Region III $\left(u < 0, v \geq 0\right)$ is obtained by the substitution $\eta = -\mu = \sin\left(v\theta\left(v\right)\right)$, which will not be given.

3.2 CX - Duality leading to the Kerr Metric

Applying the following transformations to the metric (15) which describes the interaction region of the collision of impulsive and shock gravitational waves;

$$t = M\left(x - \frac{2q}{p\,(1+p)}y\right), \qquad \phi = \frac{M}{\sqrt{M^2 - a^2}}y\ , \quad \eta = \pm\frac{(M-r)}{\sqrt{M^2 - a^2}}, \quad \mu = \cos\theta, \tag{24}$$

with

$$p = \pm\frac{\sqrt{M^2 - a^2}}{M}, \qquad q = \pm\frac{a}{M}, \qquad \text{and} \qquad M^2 > a^2, \tag{25}$$

such that $p^2 + q^2 = 1$. We have the correspondence, accordingly

$$1 - p\eta = \frac{r}{M}, \qquad 1 - \eta^2 = -\frac{\tilde{\Delta}}{M^2 - a^2}, \tag{26}$$

in which $\tilde{\Delta}$ stands for the horizon function

$$\tilde{\Delta} = r^2 - 2Mr + a^2 = (r - r_-)(r - r_+). \tag{27}$$

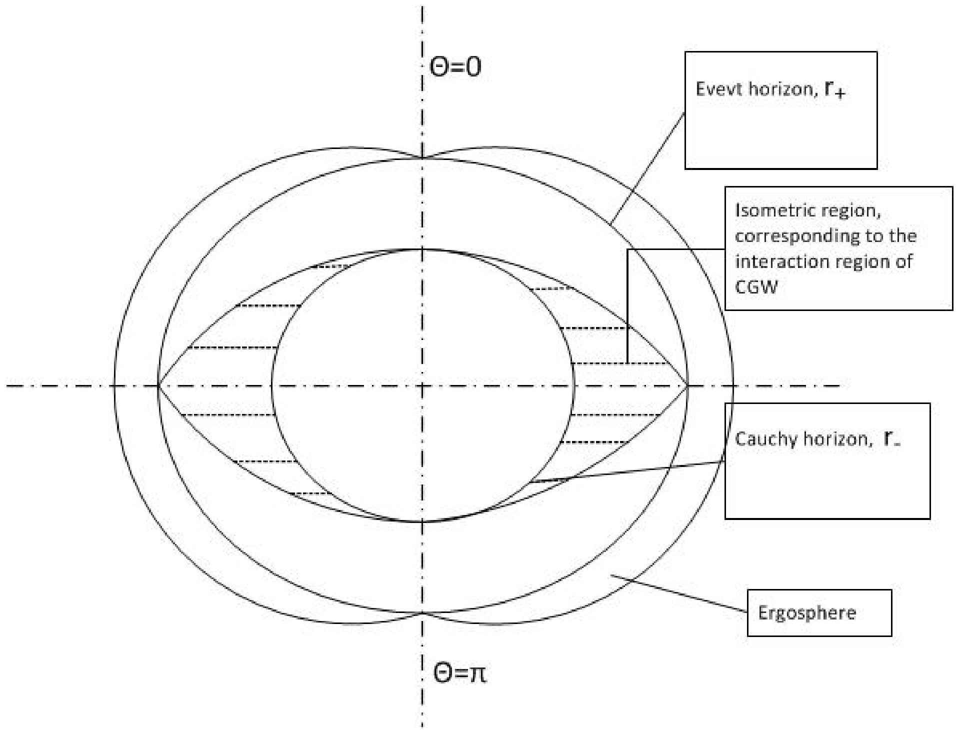
These substitutions transform the metric (15) to the following form,

$$\begin{aligned} M^2 ds^2 \;\; = \;\; & \left(\frac{\tilde{\Delta} - a^2 \sin^2\theta}{\rho^2} \right) \left[dt + \frac{2aMr \sin^2\theta}{\tilde{\Delta} - a^2 \sin^2\theta} d\phi \right]^2 \\ & - \frac{\rho^2}{\tilde{\Delta}} \left[dr^2 + \tilde{\Delta} d\theta^2 \right] - \left[\frac{\tilde{\Delta} \rho^2 \sin^2\theta}{\tilde{\Delta} - a^2 \sin^2\theta} \right] d\phi^2, \end{aligned} \tag{28}$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$, and the constants M and a represents the emergent parameters in the local isometry for mass and rotation, respectively. The roots of $\tilde{\Delta}$, namely r_+ and r_- are the event (outer) and Cauchy (inner) horizons, respectively. The only non zero Weyl scalar in these coordinates is given by

$$\Psi_2 = -\frac{M}{(r - ia\cos\theta)^3}.$$

From this transformation, we conclude that the region of interaction is locally isometric to the region in between the inner and outer horizon of the Kerr black hole. Figure 2, illustrates the region which is identical both in Kerr black hole and in the interaction region of the CGW.



Θ=0
Evevt horizon, r+
Isometric region, corresponding to the interaction region of CGW
Cauchy horizon, r-
Ergosphere
Θ=π

3.3 Timelike Kerr naked singularity

In Boyer - Lindquist coordinates (t, r, θ, ϕ) , the Kerr metric can be written as,

$$ds^2 = -\frac{\tilde{\Delta}}{\rho^2}\left[dt - a\sin^2\theta d\phi\right]^2 + \frac{\rho^2}{\tilde{\Delta}}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[adt - \left(r^2 - a^2\right)d\phi\right]^2, \tag{29}$$

If the rotational parameter dominates the mass parameter (over spinning case, $a > M$), there are no horizons and the timelike naked singularity at $r = 0$ and $\theta = \pi/2$ is developed for asymptotic observers. The metric (29) for the particular case of $M = 1$ is reduced to the following form, in the equatorial plane $\theta = \pi/2$,

$$ds^2 = -\left(1 - \frac{2}{r}\right)dt^2 - \frac{4a}{r}dtd\phi + \frac{r^2}{\bar{\Delta}}dr^2 + r^2 d\theta^2 + \left(r^2 + a^2 + \frac{2a^2}{r}\right)d\phi^2, \tag{30}$$

in which $\bar{\Delta} = r^2 + a^2 - 2r$, and the ranges of the coordinates varies as

$$0 \leq r \leq \infty, \qquad 0 \leq \theta \leq \pi, \qquad 0 \leq \phi < 2\pi. \tag{31}$$

4 Quantum Probes of Spacetime Singularities

- The scale where the singularities are forming is very small (smaller than the Planck scale), so that the classical general relativity methods in the resolution of the singularities are expected to be replaced by the quantum theory of gravity.

- Alternative methods in healing the singularities are always attracted the attentions

i String theory (G. T. Horowitz, *New J. Phys.* 201, 2005; gr-qc/0410049)

ii Loop quantum gravity (A. Ashtekar, J. Phys. Conf. Ser. 189, 2009; arXiv:0812.4703)

iii Wave probe: This method is introduced by Wald (R. M. Wald, *"Dynamics in nonglobally hyperbolic, static sapce-times"*, *J. Math. Phys. (N.Y.)* **21**, 2082 (1980)) and developed by Horowitz and Marolf (G. T. Horowitz and D. Marolf, *"Quantum probes of spacetime singularities"*, *Phys. Rev. D* **52**, 5670 (1995)).

- Horowitz-Marolf (HM) method incorporates with *self-adjoint extensions* of the spatial part of the wave operator. The classical notion of *geodesics incompleteness* with respect to point particle probe is replaced by the notion of *quantum singularity* with respect to wave probe.

4.1 Definition of Quantum Singularity

Consider a static spacetime $(M, g_{\mu\nu})$ with a timelike Killing vector field ξ^{μ}. Let t denote the Killing parameter and Σ_t denote a static slice. The Klein−Gordon equation in this space is

$$\left(\nabla^{\mu}\nabla_{\mu} - m^2\right)\psi = 0. \tag{32}$$

This equation can be written in the form

$$\frac{\partial^2\psi}{\partial t^2} = \sqrt{f}D^i\left(\sqrt{f}D_i\psi\right) - fm^2\psi = -A\psi, \tag{33}$$

in which $f = -\xi^{\mu}\xi_{\mu}$ and D_i is the spatial covariant derivative on $\mathbf{\Sigma}_t$. The Hilbert space $\mathcal{H}$, $\left(L^2\left(\mathbf{\Sigma}_t\right)\right)$ is the space of square integrable functions on $\mathbf{\Sigma}_t$. The operator A is real, positive and symmetric; therefore, its self-adjoint extensions always exist. If it has a unique extension A_E, then A is called essentially self-adjoint. Accordingly, the Klein−Gordon equation for a free particle satisfies

$$i\frac{d\psi}{dt} = \sqrt{A_E}\psi, \tag{34}$$

with the solution

$$\psi\left(t\right) = \exp\left[-it\sqrt{A_E}\right]\psi\left(0\right). \tag{35}$$

If A is not essentially self-adjoint, the future time evolution of the wave function (35) is ambiguous. Then the HM criterion defines the spacetime as quantum mechanically singular. However, if there is only a single self-adjoint extension, the operator A is said to be essentially self-adjoint and the quantum evolution described by Eq.(35) is uniquely determined by the initial conditions. According to the HM criterion, this spacetime is said to be quantum mechanically non-singular.

The problem now is to count the number of extensions of the operator A. This is done by using the concept of deficiency indices discovered by Weyl (*Math. Ann.*, **68**, 220-269, (1910).) and generelized by von Neumann (*Math. Ann.*, **102**, 49-131, (1929)) (see (*Phys. Rev. D* **60**, 104028 (1999).) for a detailed mathematical background). The determination of the deficiency indices (n_+, n_-) of the operator A, is reduced to count the number of solutions to equation

$$A\psi \pm i\psi = 0, \tag{36}$$

that belong to the Hilbert space $\mathcal{H}$. If there are no square integrable $(L^2(0,\infty))$ solutions (i.e., $n_+ = n_- = 0$) in the entire space, the operator A posessess a unique self-adjoint extension and it is called essentially self-adjoint. Consequently, the method to find a sufficient condition for the operator A to be essentially self-adjoint is to investigate the solutions satisfying equation (36) that do not belong to the Hilbert space $\mathcal{H}$.

The square integrability of the solutions of Eq.(36) for each sign $\pm$ is checked by calculating the squared norm of the solution of Eq.(36), in which the function space on each $t =$ constant hypersurface $\mathbf{\Sigma}_t$ is defined as $\mathcal{H} = \{\psi \mid \| \psi \| < \infty\}$. The squared norm can be defined as,

$$\|\psi\|^2 = \int_{\mathbf{\Sigma}_t} \sqrt{-g} g^{tt} \psi \psi^* d^3 \mathbf{\Sigma}_t. \tag{37}$$

The spatial operator A is essentially self-adjoint if neither of the solutions of Eq.(36) is square integrable over all space $L^2(0,\infty)$

4.2 Applications in Static Spacetimes

- G. T. Horowitz and D. Marolf, *Phys. Rev. D* **52**, 5670 (1995) $\rightarrow M < 0$ Schwarzchild solution, charged dilatonic black hole spacetime, fundamental string spacetimes (5- dimensional).

- A. Ishibashi and A. Hosoya, *Phys. Rev. D* **60**, 104028 (1999) $\rightarrow M < 0$ Schwarzchild solution, Reissner-Nordstrom, charged dilatonic black hole space-time, fundamental string spacetimes (5- dimensional).

- D. A. Konkowski and T. M. Helliwell, *Gen. Rel. and Grav.* **33**, 1131, (2001) $\rightarrow$ quasiregular spacetimes,

- T. M. Helliwell, D. A. Konkowski and V. Arndt, *Gen. Rel. and Grav.* **35**, 79, (2003) $\longrightarrow$ Gal'tsov - Letelier - Tod spacetimes,

- D. A. Konkowski, T. M. Helliwell and C. Wieland, *Class. Quantum Grav.* **21,** 265 (2004) $\longrightarrow$ Levi-Civita spacetimes,

- T. M. Helliwell and D. A. Konkowski, *Phys, Rev. D* **87**, 104041 (2013) $\longrightarrow$ Conformally static spacetimes,

- J. P. M. Pitelli and P. S. Letelier, *J. Math. Phys.* **48,** 092501, (2007) $\longrightarrow$ spherical and cylindrical topological defects,

- J. P. M. Pitelli and P. S. Letelier, *Phys. Rev. D* **77,** 124030 (2008) $\longrightarrow$ BTZ spacetimes,

- J. P. M. Pitelli and P. S. Letelier, *Phys. Rev. D* **80,** 104035 (2009) $\longrightarrow$ the global monopole spacetimes,

- P. S. Letelier and J. P. M. Pitelli, *Phys. Rev. D* **82,** 104046 (2010) $\longrightarrow$ cosmological spacetimes,

- O. Unver and O. Gurtug, *Phys. Rev. D* **82,** 084016 (2010) $\longrightarrow (2+1)$ $-$dimensional matter coupled black hole spacetimes.

- S. Habib Mazharimousavi, O. Gurtug and M. Halilsoy, *Int. J. Mod. Phys. D* **18,** 2061-2082, (2009) $\rightarrow$ Lovelock theory

- S. Habib Mazharimousavi, M. Halilsoy, I. Sakalli and O. Gurtug, *Class. Quant. Grav.* **27**, 105005, (2010) $\rightarrow$ Linear dilaton black hole spacetimes

- S. Habib Mazharimousavi, O. Gurtug, M. Halilsoy and O. Unver, *Phys. Rev. D* **84,** 124021 (2011) $\rightarrow$ $(2+1)$ $-$dimensional magnetically charged solution in Einstein-Power-Maxwell theory

- O. Gurtug and T. Tahamtan, *Eur. Phys. J. C* **72**, 2091 (2012) $\rightarrow$ $f(R)$ gravity

- O. Gurtug, M. Halilsoy and S. Habib Mazharimousavi, *JHEP*, 1, 178, (2014) $\longrightarrow$ $f(R)$ global monopole spacetime,

- O. Gurtug, M. Halilsoy and S. Habib Mazharimousavi, *Advances in High Energy Physics*, 684731 (2015) $\longrightarrow$ $(2+1)$ $-$dimensional power law spacetimes.

4.3 Quantum Probes of Timelike Kerr Naked Singularity

The timelike naked singularity for the Kerr metric will be probed with scalar waves satisfying the Klein-Gordon equation

$$\left(\frac{1}{\sqrt{g}}\partial_{\mu}\left[\sqrt{g}g^{\mu\nu}\partial_{\nu}\right]-\tilde{m}^{2}\right)\psi=0, \tag{38}$$

in which $\tilde{m}$ is the mass of the scalar particle. The considered model of solution to the equation (38) is called a reduced wave equation which admits solution in the form:

$$\psi\left(t,r,\theta,\phi\right)=e^{-ik\phi}f(t,r,\theta). \tag{39}$$

For the metric (30), the Klein- Gordon equation with the assumed solution can be written as

$$\frac{\partial^{2}f}{\partial t^{2}}+\frac{4aki}{r\Xi}\frac{\partial f}{\partial t}=\frac{\bar{\Delta}}{r^{2}\Xi}\left\{\frac{\partial}{\partial r}\left(\bar{\Delta}\frac{\partial}{\partial r}\right)+\frac{\partial^{2}}{\partial\theta^{2}}-\left[k^{2}\left(1-\frac{a^{2}}{\bar{\Delta}}\right)+\tilde{m}^{2}r^{2}\right]\right\}f, \tag{40}$$

where $f = f(t, r, \theta)$, k can take values of all integers and $\Xi = r^2 + a^2 + \frac{2a^2}{r}$.

Timelike naked singularity of the Kerr metric has some interesting properties that is not shared by the other naked singularities forming in static spacetimes. The Kerr naked singularity becomes visible to asymptotic observers, if one approaches to the singularity $r = 0$ from $\theta = \pi/2$ only. In other words, the surface $r = 0$, is a disc with a boundary of a ring singularity. The trajectories that approach to this surface $r = 0$ with $\theta \neq \pi/2$, do not fall into the singularity, and hence, all points are regular.

In the assumed solution in equation (39), the constant parameter k which runs for all integer values, is associated with the orbital quantum number. And, if $k = 0$ mode solution is choosen, it corresponds to *s-wave* which propagates along the equilateral plane $\theta = \pi/2$. Therefore, in order to probe the Kerr naked singularity with waves, the only wave mode that encounters with the ring singularity is the *s-wave* mode. This coincidence enables the equation (40) separable in time and spatial parts as

$$\frac{\partial^2 f}{\partial t^2} = \frac{\bar{\Delta}}{r^2 \Xi}\left\{\frac{\partial}{\partial r}\left(\bar{\Delta}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial \theta^2} - \tilde{m}^2 r^2\right\} f, \tag{41}$$

and the spatial wave operator A which will be investigated for a unique self - adjoint extensions has a form of

$$A = -\frac{\bar{\Delta}}{r^2 \Xi}\left\{\frac{\partial}{\partial r}\left(\bar{\Delta}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial \theta^2} - \tilde{m}^2 r^2\right\}. \tag{42}$$

The problem now is to count the number of extensions of the operator A.

$$A\psi \pm i\psi = 0, \tag{43}$$

If there are no square integrable ($L^2\,(0,\infty)$) solutions (i.e., $n_+ = n_- = 0$) in the entire space, the operator A posessess a unique self-adjoint extension and it is called essentially self-adjoint.

The solution to Eq.(43) is obtained by assuming the solution in separable form $\psi = R(r)\,Y(\theta)$, which yields the radial equation as

$$R'' + \frac{2(r-1)}{\bar{\Delta}}R' - \frac{1}{\bar{\Delta}}\left[r^2\left(\tilde{m}^2 \pm \frac{i\Xi}{\bar{\Delta}}\right) + c\right]R = 0, \tag{44}$$

in which prime denotes the derivative with respect to r and c is a separation constant.

The spatial operator A is essentially self-adjoint if neither of the solutions of Eq.(44) is square integrable over all space $L^2(0,\infty)$. The behavior of the Eq.(44) near $r \to 0$ and $r \to \infty$ will be considered separately.

4.3.1 The case of $r \to 0$:

In the case when $r \to 0$, the Eq.(44) simplifies to,

$$R'' - \frac{2}{a^2}R' - \frac{c}{a^2}R = 0. \tag{45}$$

If the separation constant $c > -\frac{1}{a^2}$, then the solution is

$$R(r) = e^{r/a^2}\left(C_1 e^{\alpha_1 r/a^2} + C_2 e^{-\alpha_1 r/a^2}\right), \tag{46}$$

in which $\alpha_1 = \sqrt{1 + ca^2}$. If the separation constant $c < -\frac{1}{a^2}$, then $\alpha_1 \to i\beta_1$, where β_1 is arbitrary constant and the solution is

$$R(r) = e^{r/a^2}\left(C_3 e^{i\beta_1 r/a^2} + C_4 e^{-i\beta_1 r/a^2}\right), \tag{47}$$

in which C_i $(i = 1, 2, 3, 4)$ are the integration constants.

The square integrability of the solution (46) and (47) are checked by calculating the squared norm defined in equation (37) in the limiting case of the metric (20) when $r \longrightarrow 0$, which is given by

$$\|R\|^2 \sim \int_0^{const.} r^{5/2} \, |R|^2 \, dr. \tag{48}$$

The analysis has revealed that, if the separation constant $c < -\frac{1}{a^2}$, the solution (47) is square integrable, since $\|R\|^2 < \infty$, thus, the solution belongs to the Hilbert space. But, there is a specific case for $c > -\frac{1}{a^2}$, such that if the separation constant is chosen very large, then, this specific solution fails to satisfy square integrability condition, i.e. $\|R\|^2 \rightarrow \infty$.

4.3.2 The case of $r \to \infty$:

When $r \to \infty$, the Eq.(44) reduces to

$$R'' + \frac{2}{r}R' - \left(\tilde{m}^2 \pm i\right) R = 0, \tag{49}$$

whose solution is given by

$$R(r) = \frac{1}{r}\left(C_5 \sinh\left(\alpha_2 r\right) + C_6 \cosh\left(\alpha_2 r\right)\right), \tag{50}$$

in which $\alpha_2 = \sqrt{\tilde{m}^2 \pm i}$, here C_5 and C_6 are the integration constants. The square integrability is checked with the following norm written for the case $r \to \infty$,

$$\|R\|^2 \sim \int_{const.}^{\infty} r^2 \left|R\right|^2 dr \to \infty \tag{51}$$

In view of these results; if the separation constant $c > -\frac{1}{a^2}$ and is allowed to take large values, it is shown that there is no square integrable solution to the equation (44) for the entire space. Hence, the deficiency indices $n_+ = n_- = 0$. This result indicates that for this specific wave mode solution, the spatial part of the wave operator A has a unique self-adjoint extension and the future time evolution of this specific wave becomes possible. Consequently, the occurence of timelike naked singularity in the Kerr metric remains quantum regular when probed with quantum waves obeying the Klein-Gordon equation.

5 Open Problems

- Dynamics in stationary, non-globally hyperbolic spacetimes

  i Klein-Gordon equation gives

$$\frac{\partial^2 \varphi}{\partial t^2} + B\frac{\partial \varphi}{\partial t} + A\varphi = 0 \tag{52}$$

  ii Generally, the neat division of the problem into space and time parts is not possible.

  iii Initial formulation is considered by Itai Seggev, *CQG*, 21, 2651-2668 (2004).

- Dynamics in time-dependent spacetimes.

6 Concluding Remarks

- The local isometry between a Kerr black hole and a CGW spacetime is known as the CX duality. A CH forming CGW spacetime transforms locally by a coordinate transformation into a black hole metric. Is this a coincidence ?. Can such a duality be valid for all black holes ?. In our opinion the root of such a duality traces back to the wave - particle duality and must be related to quantum gravity.

- the formation of the timelike Kerr naked singularity in the over spinning case is analysed in view of quantum mechanics with the criterion proposed by HM. This singularity is probed with waves obeying the Klein-Gordon equation. Analysis has shown that, the spatial derivative operator is essentially self-adjoint, for the waves having the separation constant $c \gg -\frac{1}{a^2}$. Hence, the classical timelike naked singularity is healed and becomes quantum regular when probed with waves described by the Klein-Gordon equation.

THANK YOU VERY MUCH FOR YOUR KIND ATTENTION