De Sitter-type spacetimes and their interpretations

Mohammad Nouri-Zonoz
University of Tehran

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Outline of the talk

1-Historical background

2-GP Transformation (Coordinates)

3-1+3 decomposition of space-times

4-Staticity condition in terms of B_g

5-Uniqueness of de Sitter-type space-times

6-Axially and cylindrically symmetric examples

M. Nouri-Zonoz, J. koohbor and H. Ramezani-Aval, PRD,91, 2015

History of the story

1917: After Einstein's static Universe, de Sitter solution was the second cosmological model incorporating a new constant, the so called cosmological constant Λ into the original EFEs

Eddington's famous quote:

Einstein Static Universe : Matter without motion

De Sitter world : Motion without matter

Einstein static Universe and de Sitter world

- Born in the same year they had different fates
- ESU turned to a pathological case (in comoving dust frame)

$$ds^2 = -dt^2 + R^2(t)\left(d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\right)$$

$$R = R_0\,, \qquad \Lambda = 4\pi\rho = \frac{1}{R_0^2}\,, \qquad p = 0$$

- De sitter continued to stay in the stage (in CSCS)

$$ds^2 = c^2dT^2 - e^{2\sqrt{\frac{\Lambda}{3}}T}\left(dR^2 + R^2d\Omega^2\right)$$

Static and dynamic forms of de sitter

$$ds^2 = (1 - \frac{\Lambda r^2}{3})c^2 dt^2 - (1 - \frac{\Lambda r^2}{3})^{-1} dr^2 + r^2 d\Omega^2$$

$$\Gamma^i_{00} \neq 0$$

$$T = t + \frac{1}{2\sqrt{\frac{\Lambda}{3}}} \ln(1 - \frac{\Lambda}{3} r^2)$$

$$R = \frac{r}{\sqrt{1 - \frac{\Lambda}{3} r^2}} e^{-\sqrt{\frac{\Lambda}{3}} t}$$

$$ds^2 = c^2 dT^2 - e^{2\sqrt{\frac{\Lambda}{3}} T} \left(dR^2 + R^2 d\Omega^2 \right)$$

$$\Gamma^i_{00} = 0$$

Comoving Synchronous Coordinate Systems (CSCS)

In a synchronous coordinate system

$$g_{00} = 1 \text{ and } g_{0\alpha} = 0$$

Comoving frame

$$u^a = (1,0,0,0) \rightarrow u_a = (1,0,0,0)$$

Synchronous coordinate system could also be a comoving one, in which the fluid elements are at rest, only if its pressure gradient vanishes.

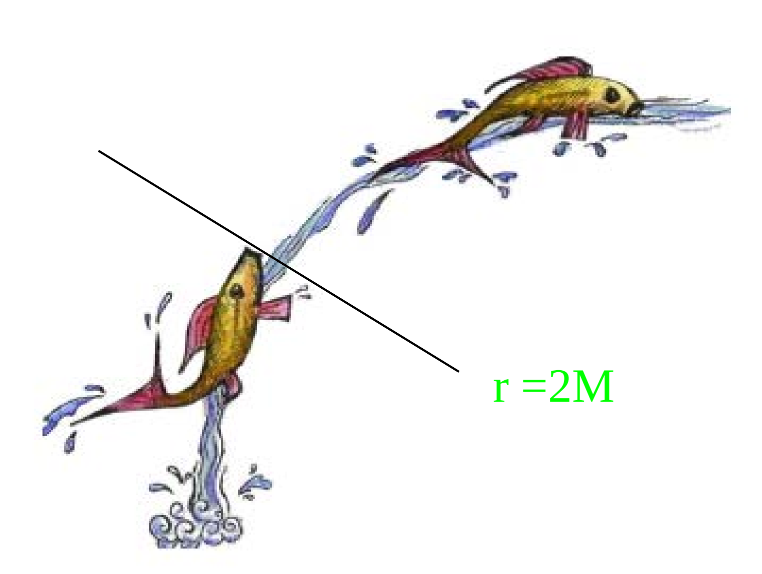
Gravitational field can not be stationary in a synchronous coordinate system.

Gullstrand-Paineleve Trax.(Coords)

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2.$$

$$T = t + \int \frac{\sqrt{1-f}}{f}\, dr$$

$$f = 1 - 2M/r$$



- Rain metric (River model of BH)

Hamilton & Lisle, AJP,2004

$$ds^2 = -dT^2 + \left(dr + \sqrt{2M/r}\, dT\right)^2 + r^2\, d\Omega^2$$

FIDOs see FFOs move radially inward with $\beta = \left(\frac{2GM}{r}\right)^{1/2}$

De Sitter in GP coordinates

$$ds^2 = (1 - \frac{\Lambda r^2}{3})c^2 dt^2 - (1 - \frac{\Lambda r^2}{3})^{-1} dr^2 + r^2 d\Omega^2$$

$$T = t + \frac{1}{2\sqrt{\frac{\Lambda}{3}}} \ln(1 - \frac{\Lambda}{3} r^2)$$

$$ds^2 = c^2 dT^2 - (dr - \sqrt{\frac{\Lambda}{3}} rc dT)^2 - r^2 d\Omega^2$$

FIDOs see FEOs move radially outward with

$$v_{esc} = \sqrt{\frac{\Lambda}{3}} rc$$

Spatial Distances and time intervals in curved spacetimes

spacetime decomposition into space and time: why?

Measurement:4d → 3d

I-A-observer B-observable

↓

II-1+3 vs 3+1 formulations

Threading and foliation

Threading vs Foliation

1+ 3:Threading GEM formalism

E_g and B_g fields

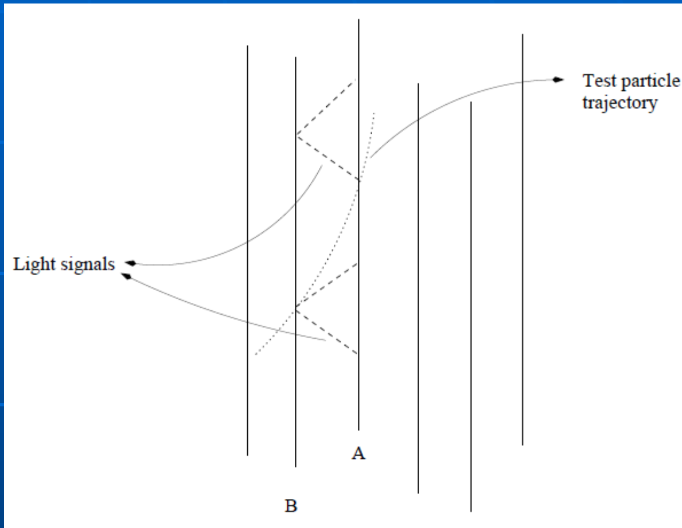
D. Lynden-Bell & M.N-Z, Rev. Mod. Phys., 1998

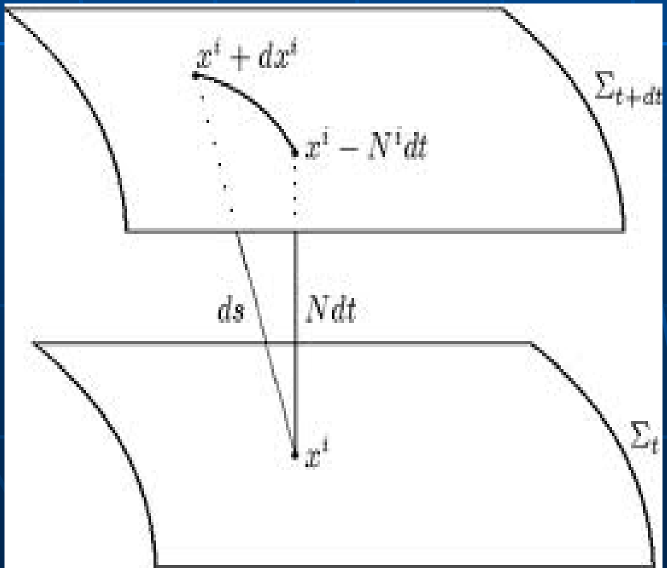
$$ds^2 = c^2 d\tau_{syn}^2 - dl^2$$

$$A_\alpha = -\frac{g_{0\alpha}}{g_{00}}$$

$$d\tau_{syn} = \frac{1}{c}\sqrt{g_{00}}\left(dx^0 - A_\alpha dx^\alpha\right)$$

$$dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta = \left(-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right)dx^\alpha dx^\beta$$





3+1:Foliation

ADM lapse and shift

$$ds^2 = -N^2 dt^2 + g_{\alpha\beta}\left(dx^\alpha + N^\alpha dt\right)\left(dx^\beta + N^\beta dt\right)$$

Indeed using this formulation, the 3-velocity of a particle in static and stationary spacetim

are given by $v^\alpha = \frac{dx^\alpha}{d\tau}$ and

$$v^\alpha = \frac{dx^\alpha}{d\tau_{syn}} = \frac{c dx^\alpha}{\sqrt{g_{00}}(dx^0 - A_\alpha dx^\alpha)},$$

respectively [3]. Now using (6) and the above definition of a test particle's 3-velocity, t

spacetime line element could be written as follows

$$ds^2 = c^2 d\tau_{syn}^2 (1 - \frac{v^2}{c^2}). \quad (1$$

The above expression gives the line element between any two nearby events in terms of t

velocity of a particle measured along its worldline (connecting the two events) in terms

the synchronized proper time which is the proper time read by clocks synchronized alo

the particle's worldline.

Also the components of the 4-velocity $u^i = \frac{dx^i}{ds}$ $(i = 0, 1, 2, 3)$ of a test particle, in terms

the components of its 3-velocity are given by

$$u^0 = \frac{1}{\sqrt{g_{00}}\sqrt{1 - v^2/c^2}} + \frac{A_\alpha v^\alpha}{\sqrt{1 - v^2/c^2}} \quad ; \quad u^\alpha = \frac{v^\alpha}{\sqrt{1 - v^2/c^2}}. \quad (1$$

Quasi-Maxwell form of EFEs for a perfect fluid

$$\nabla \times \mathbf{E}_g = 0, \quad \nabla \cdot \mathbf{B}_g = 0$$

$$\nabla \cdot \mathbf{E}_g = \frac{1}{2} h B_g^2 + E_g^2 - \frac{8\pi}{c^4} \left(\frac{p+\rho}{1-\frac{v^2}{c^2}} - \frac{\rho - p}{2} \right)$$

$$\nabla \times (\sqrt{h}\mathbf{B}_g) = 2\mathbf{E}_g \times (\sqrt{h}\mathbf{B}_g) - \frac{16\pi}{c^4} \left(\frac{p+\rho}{1-\frac{v^2}{c^2}} \right) \frac{\mathbf{v}}{c}$$

$${}^{(3)}P^{\mu\nu} = -E_g^{\mu;\nu} + \frac{1}{2} h (B_g^\mu B_g^\nu - B_g^2 \gamma^{\mu\nu}) + E_g^\mu E_g^\nu + \frac{8\pi}{c^4} \left(\frac{p+\rho}{c^2 - v^2} v^\mu v^\nu + \frac{\rho - p}{2} \gamma^{\mu\nu} \right)$$

$$T^{ab} = (p+\rho) u^a u^b - p g^{ab}$$

$$\mathbf{E}_g = -\frac{\nabla h}{2h}$$

$$\mathbf{B}_g = \nabla \times \mathbf{A}$$

Staticity condition in terms of the Gravitomagnetic field

- a stationary space-time is static if and only if its gravitomagnetic field vanishes i.e $B_g = 0$

$$x^0 \to x'^0 = x^0 - \phi(x^\alpha)$$

$$ds^2 = h(dx'^0 - A'_{g_\alpha} dx^\alpha)^2 - \gamma_{\mu\nu} dx^\mu dx^\nu$$

$$A'_{g_\alpha} = A_{g_\alpha} + \nabla_\alpha \phi$$

$$B_g^a = -\frac{1}{|\xi|}\xi^b \eta_b^{amn} \xi_{n;m} - |\xi| \xi_b \xi_n \eta^{bamn} \xi_n (\frac{1}{|\xi|^2})_{;m}$$
$$= -\frac{1}{|\xi|}\omega^a$$

$$P = -\frac{3}{2} h {B_g}^2.$$

M. N_Z & A. Parvizi, 2014

Static spacetimes in non-comoving frames

Now that we have stablished the staticity condition in terms of the non-existence of the gravitomagnetic field of the underlying stationary spacetime we are only one step away from what we mentioned as one of the the main objectives of the present study. To get there we draw the reader's attention to an intersting feature in the quasi-Maxwell form of the EFEs which is the simple fact that by equation (17),

$$\nabla \times (\sqrt{h}\mathbf{B}_g) = 2\mathbf{E}_g \times (\sqrt{h}\mathbf{B}_g) - \frac{16\pi}{c^4}\left(\frac{p+\rho}{1-\frac{v^2}{c^2}}\right)\frac{\mathbf{v}}{c}$$

a static (i.e $\mathbf{B}_g = 0$) solution produced by one element perfect fluid source, *in general* has to be in the comoving frame ($\mathbf{v} = 0$) with respect to the fluid particles, in other words in this

EXA: static interior Schwarzschild solution which is obtained in the comoving (but not synchronous) coordinate system

EXC: An obvious exception in the above feature is the case of a perfect fluid with EOS of dark energy, i.e $p = -\rho =$ const.

Irrespective of the spacetime symmetry, a perfect fluid in a non-comoving frame could be the source of a static spacetime, only if its EOS is that of dark energy/cosmological constant namely $p = -\rho = const.$.

In this way de Sitter-type solutions in their static forms are characterized as the only static, (one element) perfect fluid solutions of EFEs in non-comoving frames. To be specific, by de Sitter-type spacetimes we mean those static solutions of the generalized vacuum EFEs $R_{ab} = \Lambda g_{ab}$ $(\Lambda > 0)$, the so called (static) Einstein spaces, which reduce to the flat spacetime in the limit $\Lambda \to 0$. Further restriction to a special symmetry will lead to the static form of the corresponding de Sitter-type spacetime, for example in the case of spherical symmetry one arrives at (1) and in the case of axial or cylindrical symmetry to genuinly different solutions discussed later in the text. To achive this goal we will employ a formulation of

Application of the time transformation transforms the metric to the GP coordinates corresponding to the proper time of freely escaping observers along the outgoing radial timelike geodesics which also represent the trajectories of the fluid elements. This is so because this transformation leads to the coordinate system in which u^a = (1, 0, 0, 0), while the radial coordinate transformation takes the metric to its synchronous form in CSCS (i.e g00 = 1 and g0α = 0) where now u_a = (1, 0, 0, 0). On the other hand due to the vanishing of the pressure gradient for the perfect fluid with EOS p = −ρ = const., a synchronous coordinate system could also be a comoving one in which the perfect fluid elements are at rest.

$$T = t + \frac{1}{2\sqrt{\frac{\Lambda}{3}}} \ln(1 - \frac{\Lambda}{3} r^2)$$

$$R = \frac{r}{\sqrt{1 - \frac{\Lambda}{3} r^2}} e^{-\sqrt{\frac{\Lambda}{3}} t}$$

Nariai spacetime as an axially expanding Universe

As another example of a de Sitter-type spacetime we consider the Nariai metric which is given by the following line element in isotropic *non-comoving* spherical coordinates [12],

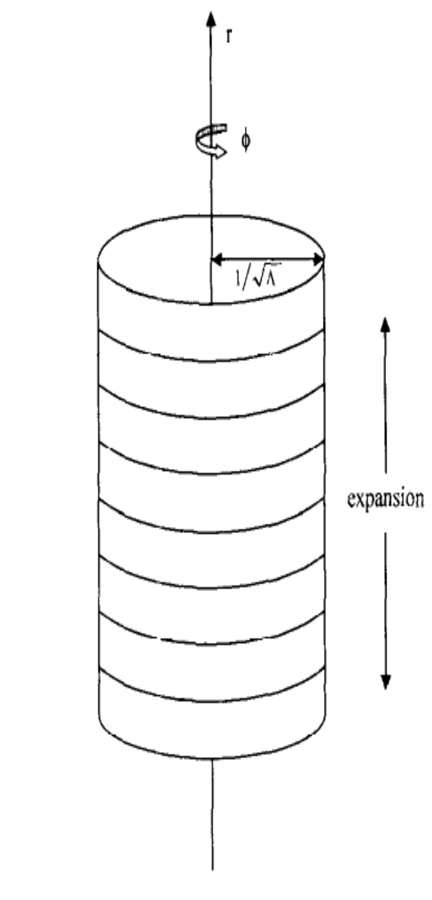
$$ds^2 = (1 - \Lambda r^2)c^2 dt^2 - (1 - \Lambda r^2)^{-1} dr^2 - \frac{1}{\Lambda}(d\theta^2 + \sin^2\theta d\phi^2), \quad (\Lambda > 0) \tag{24}$$

which is a product space $dS_2 \times S^2$. In the CSCS, its dynamical version (also called Berttoti-Kasner space) is given by the following line element

$$ds^2 = c^2 dT^2 - e^{2\sqrt{\Lambda}T} dR^2 - \frac{1}{\Lambda}(d\theta^2 + \sin^2\theta d\phi^2). \qquad dS_2 \times S^2 \tag{25}$$

$$ds^2 = (1 - \Lambda z^2)c^2 dt^2 - (1 - \Lambda z^2)^{-1} dz^2 - \frac{1}{\Lambda}(d\rho^2 + \sin^2 \rho d\phi^2)$$

$$ds^2 = (1 - \Lambda z^2)c^2 dt^2 - (1 - \Lambda z^2)^{-1} dz^2 - \frac{1}{(1 + \frac{\Lambda}{4}\bar{\rho}^2)^2}(d\bar{\rho}^2 + \bar{\rho}^2 d\phi^2)$$



$$dS_2 \times R^2$$

reduces to flat spacetime as $\Lambda \to 0$.

W. Rindler, Phys. Lett. A **245**, 363 (1998)

$$T = t + \frac{1}{2\sqrt{\Lambda}} \ln(1 - \Lambda z^2)$$

$$Z = \frac{z}{\sqrt{1 - \Lambda z^2}} e^{-\sqrt{\Lambda} t},$$

↓

$$ds^2 = c^2 dT^2 - e^{2\sqrt{\Lambda} T} dZ^2 - \frac{1}{(1 + \frac{\Lambda}{4}\bar{\rho}^2)^2} (d\bar{\rho}^2 + \bar{\rho}^2 d\phi^2)$$

On the other hand, its one-directional expansion and one-directional field are rather puzzling in light of the fact that a Λ-term in the field equations is tantamount to thc energy tensor of an exotic but isotropic fluid. How can isotropic sources "cause" a one-directional field? Is this another example of an anti-Marchian universe, i.e. one whose spacetime symmetries are incompatible with the symmetries of its sources? The answer

How can isotropic sources "cause" a one-directional field? Is this another example of an anti-Machian universe, i.e. one whose spacetime symmetries are incompatible with the symmetries of its source?

By the above arguments our answer to this question is clear. The spacetime symmetries are compatible with the symmetries of its both dark and nondark sources. In this case while there is no nondark source for the field, it possesses a dark source which is a perfect fluid with an EOS $p = -\rho$ and a unidirectional (bulk) motion. This motion defines a distinct CSCS in which the dynamic version of the metric is given by (26) and whose unidirectional expansion is naturally dictated by the dark fluid's velocity.

The class of static cylindrically symmetric vacuum solutions, which was found by Levi-Civita in 1919 [1], can be written in the form

$$ds^2 = -\rho^{4\sigma/\Sigma}dt^2 + \rho^{-4\sigma(1-2\sigma)/\Sigma}dz^2 + C^2\rho^{2(1-2\sigma)/\Sigma}d\phi^2 + d\rho^2, \qquad (1)$$

where $\Sigma = 1 - 2\sigma + 4\sigma^2$. The parameter $\sigma \in (0, \frac{1}{4})$ may be interpreted as the mass per unit length of the source located along the axis $\rho = 0$, while C is the conicity parameter (see [2] for more details). When $\sigma = 0$, Minkowski space in cylindrical coordinates is recovered.

In 1986, a generalisation of (1) to include a non-zero cosmological constant Λ was obtained by Linet [3] and Tian [4] (see also [5]) in the form

$$ds^2 = Q^{2/3}\left(-P^{-2(1-8\sigma+4\sigma^2)/3\Sigma}\,dt^2 + P^{-2(1+4\sigma-8\sigma^2)/3\Sigma}\,dz^2 + C^2P^{4(1-2\sigma-2\sigma^2)/3\Sigma}\,d\phi^2\right) + d\rho^2, \quad (2)$$

where ρ is a proper radial distance from the axis and

$$Q(\rho) = \frac{1}{\sqrt{3\Lambda}}\sin\left(\sqrt{3\Lambda}\,\rho\right), \qquad P(\rho) = \frac{2}{\sqrt{3\Lambda}}\tan\left(\frac{\sqrt{3\Lambda}}{2}\,\rho\right). \qquad (3)$$

J. B. Griffiths and J. Podolsky, Phys. Rev. D **81**, 064015 (2010).

Interestingly, the "no source" limit $\sigma = 0$ *is not the (anti-)de Sitter space*, as one would naturally expect! For $\sigma = 0$, the Linet–Tian metric reduces to

$$ds^2 = p^2(-dt^2 + B^2\,d\psi^2) + \frac{4C^2}{3\Lambda}\frac{(1-p^3)}{p}\,d\phi^2 + \frac{3}{\Lambda}\frac{p}{(1-p^3)}\,dp^2, \qquad (15)$$

VI. CYLINDRICALLY SYMMETRIC DE SITTER-TYPE SPACETIME

To reinforce the above interpretation and as another example of a de Sitter-type spacetime we consider the following static solution to the MEFEs,

$$ds^2 = \cos^{4/3}\left(\frac{\sqrt{3\Lambda}}{2}\rho\right)(dt^2 - dz^2) - d\rho^2 - \frac{4}{3\Lambda}\sin^2\left(\frac{\sqrt{3\Lambda}}{2}\rho\right)\cos^{-2/3}\left(\frac{\sqrt{3\Lambda}}{2}\rho\right)d\phi^2. \quad (31)$$

which could be obtained from the spacetime metric of a cylindrical distribution of matter in the presence of the cosmological constant, by setting the linear mass density equal to zero

reduces to flat spacetime as $\Lambda \to 0$.

$$K \equiv R^{abcd}R_{abcd} = \frac{4}{3}\frac{\left(2\cos^4\left(\frac{\sqrt{3\Lambda}}{2}\rho\right)+1\right)}{\cos^4\left(\frac{\sqrt{3\Lambda}}{2}\rho\right)}\Lambda^2 = \frac{8}{3}\Lambda^2 + \frac{4}{3}\Lambda^2\cos^{-4}\left(\frac{\sqrt{3\Lambda}}{2}\rho\right)$$

above solution can be obtained as a special case ($\gamma = -1$) of static cylindrically symmetric perfect fluid solutions with barotropic EOS $p = \gamma\rho$ [22] in the following equivalent form,

$$ds^2 = F^{2/3}(dt^2 - dz^2) - F^{-1}d\bar{\rho}^2 - F^{-1/3}\bar{\rho}^2 d\phi^2 \tag{33}$$

$$F = 1 - \frac{3}{4}\Lambda\bar{\rho}^2, \tag{34}$$

by the following transformation

$$\bar{\rho} = \frac{2}{\sqrt{3\Lambda}}\sin\left(\frac{\sqrt{3\Lambda}}{2}\rho\right). \tag{35}$$

$$ds^2 = dT^2 - F^{-1/3}(\tilde{\rho},T)B(\tilde{\rho},T)(d\tilde{\rho}^2 + \tilde{\rho}^2 d\phi^2) - F^{2/3}(\tilde{\rho},T)dz^2 \tag{36}$$

through the transformations

$$dt = F^{-2/3}dT + ABF^{-1}d\tilde{\rho} \tag{37}$$

$$d\bar{\rho} = A(\tilde{\rho},T)dT + B(\tilde{\rho},T)d\tilde{\rho}, \tag{38}$$

where the functions A and B are given by

$$A = (F^{1/3} - F)^{1/2} \tag{39}$$

$$\int \frac{dB}{\sqrt{(1 - \frac{3}{4}\Lambda\tilde{\rho}^2 B^2)^{1/3} - (1 - \frac{3}{4}\Lambda\tilde{\rho}^2 B^2)}} = \frac{T}{\tilde{\rho}}. \tag{40}$$

Conclusions

A one element perfect fluid in a non-comoving frame could be the source of a static spacetime, only if its EOS is that of dark energy.

The above assertion shows that the de Sitter spacetime and de Sitter-type solutions are unique solutions, and further implies why there should be different de Sitter-type spacetimes.

Why, in the absence of matter, there is a unique flat spacetime solutuion of EFEs but genuinely different de Sitter-type solutions of MEFEs?

Our answer :

There is a hidden parameter which distinguishes between diffrent de Sitter-type solutions and that is the velocity of the perfect fluid with EOS $p = -\rho = const.$ which formally plays the role of the cosmological constant in these solutions.

$$T^{ab} = (p + \rho)u^a u^b - p g^{ab}$$

Borrowing Eddington's language:

Einstein static Universe is a Universe in a coordinate system with "comoving matter and non-comoving dark energy",

Time-(in)dependent de Sitter-type spacetimes are Universes in a coordinate system with "(non-) comoving dark fluid"

and

Minkowski spacetime is the "no matter no dark energy" or "no nothing" Universe

Identification of the geometric (cosmological constant) term Λgij, with a perfect fluid with the EOS $p = -\rho = const.$, although mathematically consistent , obscures the crucial role of the dark fluid's velocity in defining a preferred comoving coordinate system in de Sitter-type spacetime

so

For a consistent interpretation of de Sitter-type spacetimes one should model the cosmological term as a perfect (dark) fluid.

Thanks for your attention