Non-linear Fluctuations In Anomalous Hydrodynamic

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Work in Progress in Collaboration with Ali Davody

IPM

1- Hydrodynamics

Microscopic Conservation Laws:

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\mu J^\mu = 0$$

Local Thermal Equilibrium:

Hydro variables: $\{\, u^\mu(x), \quad T(x), \quad \mu(x) \,\}$

2- Hydrodynamic Expansion:

In the long-wavelength limit:

$$T^{\mu\nu}(x) = T^{\mu\nu}_{(0)}(x) + T^{\mu\nu}_{(1)}(x) + \ldots$$

$O(\partial^0)$ $\quad$ $O(\partial)$

$$J^{\mu}(x) = J^{\mu}_{(0)}(x) + J^{\mu}_{(1)}(x) + \ldots$$

3- Constitutive Relations in LL Frame:

Landau-Lifshitz Frame:

$$T^{\mu\nu}\, u_\nu = \epsilon\, u^\mu$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} - \eta\, \Delta^{\mu\alpha}\Delta^{\nu\beta}\Big(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\partial_\mu u^\mu\Big) - \zeta\Delta^{\mu\nu}\partial_\lambda u^\lambda + O(\partial^2)\,,$$

$$J^\mu = nu^\mu - \sigma T\Delta^{\mu\nu}\partial_\nu(\mu/T) + \chi_{\rm T}\Delta^{\mu\nu}\partial_\nu T + O(\partial^2)\,.$$

$$\Delta^{\mu\nu} \equiv \eta^{\mu\nu} + u^\mu u^\nu$$

Constraints From the 2d Law of Thermo:

$$\partial_\mu S^\mu \geqslant 0 \longrightarrow S^\mu = s u^\mu + (\text{gradient corrections})$$

$$\Rightarrow \quad \eta \geqslant 0\,, \quad \zeta \geqslant 0\,, \quad \sigma \geqslant 0\,, \quad \chi_{\rm T} = 0$$

4- Parity Violating Fluid:

Long time missed term!

D. T. Son and P. Surowka
Phys.Rev.Lett. 103 (2009)

Vorticity term: $$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho.$$

$$J^\mu = n u^\mu - \sigma T \Delta^{\mu\nu} \partial_\nu (\mu/T) + \xi \omega^\mu$$

Firstly found through investigating hydrodynamics of charged black holes

N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam, and P. Surówka, arXiv:0809.2596 [hep-th].

5- Hydrodynamics with Anomalies:

In the presence of an external magnetic field:

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda \,,$$

$$\partial_\mu j^\mu = C E^\mu B_\mu$$

$$E^\mu = F^{\mu\nu} u_\nu, \; B^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$$

Corrected current:

$$J^\mu = n u^\mu - \sigma T \Delta^{\mu\nu} \partial_\nu (\mu/T) + \sigma E^\mu + \xi \omega^\mu + \xi_B B^\mu$$

CMV and CME coefficients:

$$\xi = C\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + P}\right), \quad \xi_B = C\left(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + P}\right)$$

6- Hydrodynamics Fluctuations:

Fluctuations around: $v^i = 0,\ T = \text{const},\ \mu = 0$

Fluctuating fields:

$$\begin{aligned} \delta\epsilon(t,\mathbf{x}) &= \delta T^{00} \\ \pi_i(t,\mathbf{x}) &= T^{0i} \\ n(t,\mathbf{x}) &= J^0 \end{aligned} \quad \longleftrightarrow \quad \varphi_a$$

Correlation Functions

$$G^R_{ab}(t-t',\mathbf{x}-\mathbf{x}') \equiv -i\theta(t-t')\,\langle[\varphi_a(t,\mathbf{x}),\varphi_b(t',\mathbf{x}')]\rangle$$

7- Retarded Green's Functions:

Equation of motion linearized:

$$\partial_t \varphi_a(t, \mathbf{k}) + M_{ab}(\mathbf{k}) \varphi_b(t, \mathbf{k}) = 0$$

Linear Response Theory:

L. P. Kadanoff and P. C. Martin
Ann. Phys. **24** (1963) 419.

$$G^R_{ab}(\omega, \mathbf{k}) = -(\delta_{ac} + i\omega K_{ac}(\mathbf{k}))\, \chi_{cb}$$

$$K_{ab}(\mathbf{k}) = -i\omega\delta_{ab} + M_{ab}(\mathbf{k})$$

$$\chi_{ab} = \left(\frac{\partial \varphi_a}{\partial \lambda_b}\right)$$

$$\delta H = -\int d^d x\, \lambda_a(t, \mathbf{x})\, \varphi_a(t, \mathbf{x})$$

8- Fluid in a Background Magnetic Field:

Thermodynamic Solution:

$$v^i = 0,\ T = \text{const},\ \mu = 0,\ B = \text{const}$$

Equation of motion linearized:

$$\partial_t \delta n + k^2 D \delta n + \frac{i}{\bar{w}} k_j F^{jk} \pi_k = 0$$

$$\partial_t \delta \epsilon + i k_j \pi^j = 0$$

$$\partial_t \pi^j + i k_j (\beta_1 \delta\epsilon + \beta_2 \delta n) + \gamma_s k_j (k.\pi) + \gamma_\eta (k^2 \pi_j - k_j (k.\pi)) + i D F^{jm} k_m \delta n = \frac{\sigma}{\bar{w}} F^{jk} F_{kn}$$

$$\eta = -\frac{\omega}{\mathbf{k}^2} \frac{1}{d-1} \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \right) \operatorname{Im} G^R_{\pi_i \pi_j}(\omega, \mathbf{k} \to 0)$$

9- Computations in a Specific Frame:

$$\vec{B} = (0, 0, B)$$
$$\vec{k} = (0, k_y, k_z)$$

Fileds and their Sources:

$$\phi_a = (\delta\epsilon, \pi_x, \pi_y, \pi_z, \delta n)$$

$$\lambda_a = (\frac{\delta T}{T}, v_x, v_y, v_z, \delta\mu - \frac{\mu}{T}\delta T)$$

$$M_{ab} = \begin{pmatrix} 0 & 0 & ik_y & ik_z & 0 \\ 0 & \gamma_\eta k^2 + \frac{\sigma}{\bar{w}}B^2 & 0 & 0 & -iDBk_y \\ i\beta_1 k_y & 0 & \gamma_s k_y^2 + \gamma_\eta k_z^2 + \frac{\sigma}{\bar{w}}B^2 & (\gamma_s - \gamma_\eta)k_y k_z & ik_y B \\ 0 & 0 & (\gamma_s - \gamma_\eta)k_y k_z & \gamma_s k_z^2 + \gamma_\eta k_y^2 & ik_z B \\ 0 & \frac{i}{\bar{w}}Bk_y & 0 & 0 & k^2 D \end{pmatrix}$$

$$\chi_{ab} = \begin{pmatrix} T\left(\frac{\partial\epsilon}{\partial T}\right)_{\mu/T} & 0 & 0 & 0 & \left(\frac{\partial\epsilon}{\partial\mu}\right)_T \\ 0 & \bar{w} & 0 & 0 & 0 \\ 0 & 0 & \bar{w} & 0 & 0 \\ 0 & 0 & 0 & \bar{w} & 0 \\ T\left(\frac{\partial n}{\partial T}\right)_{\mu/T} & 0 & 0 & 0 & \left(\frac{\partial n}{\partial\mu}\right)_T \end{pmatrix}$$

10- Hydrodynamics Modes:

Retarded Green's Functions:

$$G^R_{\epsilon\epsilon}(\omega,\mathbf{k}) = \frac{\bar{w}\,\mathbf{k}^2}{\omega^2 - v_s^2 k^2 - (\omega_1(\mathbf{k}) + \omega_2(\mathbf{k}))\,\omega}$$

$$G^R_{nn}(\omega,\mathbf{k}) = \frac{-i\sigma\,\mathbf{k}^2\left(\omega + i\mathbf{k}^2\gamma_\eta + i(\sigma - 1 + \hat{\mathbf{B}}.\hat{\mathbf{k}})\mathbf{B}^2/\bar{\omega}\right)}{\omega^2 - (\omega_3(\mathbf{k}) + \omega_4(\mathbf{k}))\,\omega}$$

modified sound modes:

$$\omega_{1,2} = \pm v_s k - i\frac{\gamma_s}{2}k^2 - i\frac{\sigma}{2\bar{w}}\mathbf{B}^2(1 - \hat{\mathbf{B}}.\hat{\mathbf{k}})$$

modified diffusive modes:

$$\omega_{3,4} = \frac{i}{2}\left(-\mathbf{k}^2(D + \gamma_\eta) - \sigma\mathbf{B}^2/\bar{\omega} \pm \sqrt{(\mathbf{k}^2(D - \gamma_\eta) + \mathbf{B}^2\sigma/\bar{\omega})^2 + 4\mathbf{B}^2(\mathbf{k}^2 - (\hat{\mathbf{B}}.\mathbf{k})^2)}\right)$$

$$\omega_5 = \frac{-i}{2\bar{\omega}}\left(2\mathbf{k}^2\gamma_\eta - i\xi_\omega\mathbf{B}.\mathbf{k} + 2\mathbf{B}^2\,(\hat{\mathbf{B}}.\hat{\mathbf{k}})\,\sigma\right)$$

11- $B \to 0$ Limit:

$$\omega_1(\mathbf{k}) \to v_s k - i\frac{\gamma_s}{2}k^2$$
$$\omega_2(\mathbf{k}) \to -v_s k - i\frac{\gamma_s}{2}k^2$$
$$\omega_3(\mathbf{k}) \to -iD\mathbf{k}^2$$
$$\omega_4(\mathbf{k}) \to -i\gamma_\eta \mathbf{k}^2$$
$$\omega_5(\mathbf{k}) \to -i\gamma_\eta \mathbf{k}^2$$

Rapid equilibriating at linear order:

$$G_{\epsilon\epsilon}(t, \boldsymbol{k}) = e^{-\frac{1}{2}\boldsymbol{k}^2\gamma_s|t|} \cos(|\boldsymbol{k}|v_s t)\, C$$

Long Time Tail at Non-linear level:

$$\mathcal{V}^{-1}\left\langle \tfrac{1}{2}\{J_a^i(t), J_a^j(0)\}\right\rangle = \frac{T}{\bar{w}}\frac{\delta^{ij}}{12}\left\{\frac{1}{[(D+\gamma_\eta)\,\pi|t|]^{3/2}}\,\chi\right\} + (\text{exponential decay})$$

12- Work in Progress:

1. **Green's function in the presence of anomalies**
2. **Kubo formula for the transport coefficients**
3. **Contribution of non-linear terms to transport coefficients**

Thank You