Holographic Entanglement Entropy

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IPM School and Workshop on Applied AdS/CFT

Feb 28- March 4, 2015, Tehran

Plan

- Review of (holographic) entanglement entropy

- Explicit example: static and time dependent holographic entanglement entropy

- Special topics: different lines of research

References

P. Calabrese and J. Cardy, "Entanglement entropy and conformal field theory," arXiv:0905.4013.

T. Nishioka, S. Ryu and T. Takayanagi, "Holographic Entanglement Entropy: An Overview," arXiv: 0905.0932.

S. N. Solodukhin, "Entanglement entropy of black holes," arXiv:1104.3712

H. Liu and S. J. Suh, "Entanglement growth during thermalization in holographic systems," arXiv:1311.1200

Entanglement entropy

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

The reduced density matrix of the subsystem A

$$\rho_A = \text{Tr}_B(\rho)$$

Then the entanglement entropy is defined as the von-Neumann entropy

$$S_A = -\text{Tr}(\rho_A \ln \rho_A)$$

A measure how much a given quantum state is quantum mechanically entangled.

Properties of Entanglement entropy

1. For pure state $S_A = S_B$

2. For two subspace A and B, the strong subadditivity is

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$$

3. Leading divergence term is proportional to the area of the boundary ∂A

$$S_A = c_0 \frac{\text{Area}(\partial A)}{\epsilon^{d-1}} + O(\epsilon^{-(d-2)}),$$

3. Area law?

We have seen that the entanglement entropy is proportional to the area of the entangling region. How general is this?

Already in two dimensions the entanglement entropy is proportional to the log

$$S = \frac{c}{3} \ln \frac{\ell}{\epsilon}$$

It is possible to have other behavior. In particular for the case where the corresponding theory is non-local.

Let us explore it in an explicit example.

General solution with hyperscaling factor

$$S = -\frac{1}{16\pi G_N}\int d^{D+2}x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{N_g} e^{\lambda_i\phi}F^{(i)^2}\right],$$

where $V(\phi) = V_0 e^{\gamma\phi}$, G is the Newton constant, γ, V_0 and λ_i are free parameters of the model.

One of the gauge field is required to produce an anisotropy while the above particular form of the potential is needed to get hyperscaling violating factor. The other gauge fields make the background charged. In what follows we will consider $N_g = 2$.

The model admits solutions with hyperscaling violating factor

$$ds^2 = r^{-2\frac{\theta}{D}}\Big(-r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2\Big),$$

Under scaling

$$t \to \xi^z t, \qquad x_i \to \xi x, \qquad r \to \xi^{-1} r$$

the metric scales $ds \to \xi^{\theta/D} ds$.

$$S \sim T^{(D-\theta)/z}$$

It has exact charged black hole solutions as follows

$$ds^2 = r^{-2\frac{\theta}{D}}\left(-r^{2z}f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}^2\right), \quad \phi = \beta \ln r,$$

$$A_t^{(1)} = \sqrt{\frac{2(z-1)}{D-\theta+z}}\, r^{D-\theta+z}, \qquad A_t^{(2)} = \sqrt{\frac{2(D-\theta)}{D-\theta+z-2}}\,\frac{Q}{r^{D-\theta+z-2}},$$

with $\beta = \sqrt{2(D-\theta)(z-1-\theta/D)}$ and

$$f(r) = 1 - \frac{m}{r^{D-\theta+z}} + \frac{Q^2}{r^{2(D-\theta+z-1)}}.$$

where z is the dynamical exponent and θ is the hyperscaling violation exponent.

To be more concrete, consider $m = Q = 0$ and after a double Wick rotation as follows

$$t \to iy, \qquad x_d \to it,$$

one gets

$$ds^2_{d+2} = r^{\frac{2\theta}{d}} \left(\frac{dy^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{\sum_{i=1}^{d-1} dx_i^2}{r^2} - \frac{dt^2}{r^2} \right).$$

Let us compute the holographic entanglement entropy for the following strip

$$\frac{\ell}{2} \leq y \leq \frac{\ell}{2}, \qquad 0 \leq x_i \leq L, \ \text{for } i = 1, \cdots, d-1.$$

Setting $y = y(r)$ the induced metric of the co-dimension two hyper surface is

$$ds^2_{ind} = r^{2\frac{\theta}{d}} \left[\left(\frac{y'^2}{r^{2z}} + \frac{1}{r^2} \right) dr^2 + \frac{\sum_{i=1}^{d-1} dx_i^2}{r^2} \right].$$

Therefore the area of the surface is

$$A = L^{d-1} \int_\epsilon dr \, \frac{\sqrt{r^{2(z-1)} + y'^2}}{r^{d+z-\theta-1}}.$$

Minimizing this area, for general θ, d and z, one finds

$$S = \frac{L^{d-1}}{4(d-\theta-1)G_N} \left(\frac{1}{\epsilon^{d-\theta-1}} - b_0 \frac{c_0^{(d-\theta-1)/z}}{\ell^{(d-\theta-1)/z}} \right).$$

For $\theta = d - 1$

$$S = \frac{1}{4z\pi G_N} \frac{L^{d-1}}{r_F^{d-1}} \ln \frac{z\ell}{\epsilon^z},$$

For $\theta = d$

$$S \sim L^{d-1} \ell^{1/z}.$$

For $z = 1$ it is indeed a volume law!

The properties of the system may be reflected in the behavior of the holographic Entanglement. May be used as a probe: Different phase transitions, Fermi surface,....

4. Higher derivative

The holographic formula we have considered is for Einstein gravity. Motivated by the Wald formula it is interesting to see how this formula is modified in the presence of higher derivative corrections to Einstein gravity.

Unlike the Wald formula for black hole entropy there is no a rigorous derivation for a general expression when we have arbitrary higher derivative corrections.

Consider an action with R^2 terms

$$S = -\frac{1}{16G_N}\int d^{d+2}x\sqrt{g}\left[R - 2\Lambda + (\alpha R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^2)\right],$$

For the Gauss-Bonnet gravity where $\alpha = \lambda, \beta = -4\lambda, \gamma = \lambda$, the holographic entanglement entropy is argued to be

$$S_A = \mathrm{Min}_{\gamma_A} \left[\frac{1}{4G_N} \int_{\gamma_A} d^d x \sqrt{h} (1 + 2\lambda R_{int}) \right],$$

where R_{int} is the intrinsic curvature of γ_A.

Nevertheless for generic case one still needs a general formula!

The main problem comes from the fact that, unlike horizon, for a generic hypersurface the extrinsic curvature is non-zero.

Therefore beyond the terms as that of Wald formula one could have other terms with is proportional to extrinsic curvature.

The corresponding entropy functional for our case becomes

$$S_A \sim \int d^2\zeta \sqrt{h} \left[\gamma R - \beta \left(R_{\mu\nu} n_i^\mu n_i^\nu - \frac{1}{2} \mathcal{K}^i \mathcal{K}_i \right) + \alpha \left(R_{\mu\nu\rho\sigma} n_i^\mu n_j^\nu n_i^\rho n_j^\sigma - \mathcal{K}^i_{\mu\nu} \mathcal{K}_i^{\mu\nu} \right) \right]$$

where $i = 1, 2$ denotes two transverse directions to a co-dimension two hyper-surface in the bulk, n_i^μ are two unit mutually orthogonal normal vectors on the co-dimension two hyper-surface and $\mathcal{K}^{(i)}$ is the trace of two extrinsic curvature tensors defined by

$$\mathcal{K}^{(i)}_{\mu\nu} = \pi^\sigma{}_\mu \pi^\rho{}_\nu \nabla_\rho (n_i)_\sigma, \qquad \text{with} \quad \pi^\sigma{}_\mu = \epsilon^\sigma{}_\mu + \xi \sum_{i=1,2} (n_i)^\sigma (n_i)_\mu$$

where $\xi = -1$ for space-like and $\xi = 1$ for time-like vectors. Moreover h is the induced metric on the hyper-surface whose coordinates are denoted by ζ.

A way to find a reasonable expression is to use the replica trick which in general leads to a singular geometry. Then one should extract the contribution of the cone!

Near the cone the metric may be written as

$$ds^2 = g(r)d\tau^2 + dr^2 + \gamma_{ij}(r,x)dx^i dx^j \qquad g(r) \sim r^2 + \mathcal{O}(r^4)$$

with the identification $\tau \equiv \tau + 2\pi n$.

One may regularized the cone

$$ds^2 = e^{2\sigma(x,r)}[d\tau^2 + f_n(r)dr^2 + \gamma_{ij}(r,\tau,x)dx^i dx^j, \qquad f_n(r) = \frac{r^2 + b^2 n^2}{r^2 + b^2}$$

$$\gamma(r,\tau,x) = h_{ij}(x) + 2K^a_{ij} n^a r^n + g_{ij}(x) r^2 + (K^a K^b)_{ij} n^a n^b r^{2n} + \cdots$$

Other regularizations may be used: The results should be independent of the regularization.

Using this metric one can find the contribution of each rearm . For example

$$R^{(n)}_{\mu\nu} = R^{\text{reg}}_{\mu\nu} + 2\pi n^a_\mu n^b_\nu \delta_\Sigma$$

which leads to a term in the entropy as follows

$$S_A \to \int_\Sigma R_{\mu\nu} n^\mu_a n^\nu_b$$

On the other hand from the extrinsic curvature one gets

$$S_A \to -\frac{1}{2}\int_\Sigma K^2$$

So one arrives that

$$\int R_{\mu\nu}R^{\mu\nu} \to \int_\Sigma (R_{\mu\nu} n^\mu_a n^\nu_b - \frac{1}{2}K^2)$$

Other terms may be computed in the same way.

Is this the right thing to do? What about the regularization?

Let us check it for 4D conformal gravity

$$\begin{aligned} S &= -\frac{\kappa}{32\pi}\int d^4x\,\sqrt{-g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) \\ &= -\frac{\kappa}{32\pi}\mathsf{GB}_4 - \frac{\kappa}{16\pi}\int d^4x\,\sqrt{-g}\,\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) \\ &= -\frac{\kappa}{32\pi}\mathsf{GB}_4 + S^{\mathsf{dyn}}, \end{aligned}$$

where GB_4 is the four dimensional Gauss-Bonnet action which is a total derivative and does not contribute to the equations of motion. Note that since the Gauss-Bonnet term is topological, the whole dynamics must be encoded in the second term (S^{dyn}).

It is then easy to compute the entanglement entropy (for example for and AdS solution)

$$S_{EE}^{\text{dyn}} = \kappa L_y \left[\frac{1}{\epsilon} - \frac{2\pi \Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} \frac{1}{\ell} \right].$$

Going through the same procedure for the Gauss-Bonnet term, one arrives at

$$S_{EE}^{\text{GB}} = \kappa L_y \left(-\frac{1}{\epsilon} \right).$$

It is then clear that taking both contributions into account the divergent term will drop leading to a finite entanglement entropy.

More over S_{EE}^{dyn} is the same as that of Einstein gravity. It is consistent with the relation between 4D CG and 4D Einstein gravity.

For higher order derivatives more works are needed.