بسم الله الرحمن الرحیم

Longitudinal Beam Dynamics

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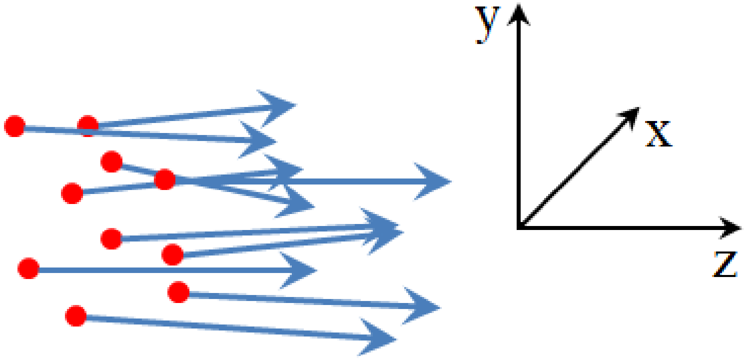
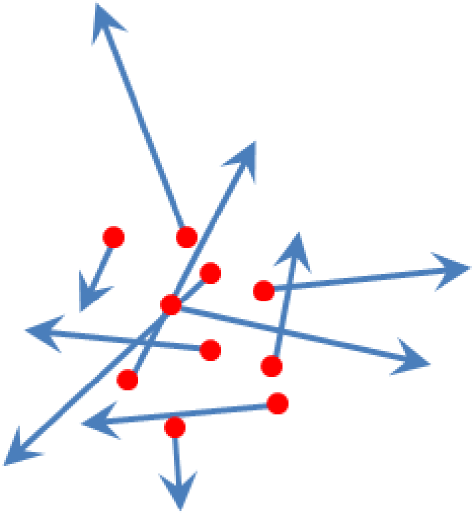
Contents

1. Introduction

Charged particle beams

- ➢ Particles traveling in nearly the same direction with nearly the same energy.
  - ✓ $\sigma_{E_k} \ll E_k$
  - ✓ $p_x \, , p_y \ll p_z$





Longitudinal direction & Transverse Plane

- ➢ Independent motion. ⟹ 3D⟶2D+1D
- ➢ Acceleration vs Beam Control.

The goal of a Beam Dynamics study

- To study the beam behavior under the influence of electromagnetic fields of accelerator components (magnets and cavities and …) and the beam itself.
    - ✓ Maxwell's equations: $\begin{cases} \boldsymbol{\nabla}.\boldsymbol{E} = \rho/\varepsilon_0 \\ \boldsymbol{\nabla}.\boldsymbol{B} = 0 \\ \boldsymbol{\nabla}\times\boldsymbol{E} = -\,\partial\boldsymbol{B}/\partial t \\ \boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0(J + \partial\boldsymbol{E}/\partial t) \end{cases}$
    - ✓ Lorentz force: $d\boldsymbol{P}/dt = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$

- Determination of the required array of the electromagnetic fields (specification of the accelerator components).

$\Longrightarrow$ Beam Dynamics Design

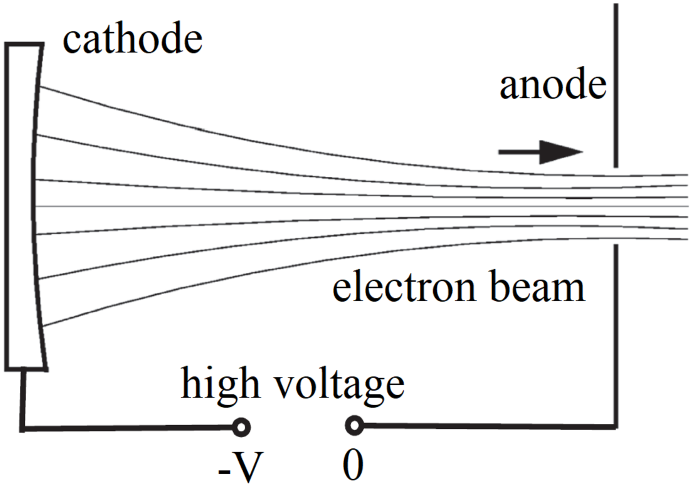
Perquisites

- ✓ Classical mechanics (including Hamiltonian dynamics)
- ✓ Electromagnetic theory.
- ✓ Special relativity.
- ✓ Statistical mechanics (only for Space charge dynamics)

2. Electrostatic vs RF acceleration

Electrostatic acceleration limitations

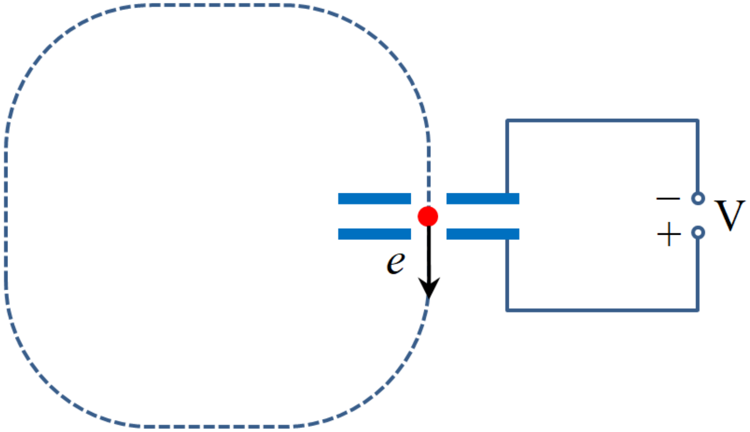
- Breakdown effect
  - ✓ Highest voltage ever: ~12 MV
  - ✓ Large and expensive machines.



Holifield Heavy Ion Accelerator, ORNL

Electrostatic acceleration limitations

- Non-repeatable
  - ✓ Electrostatic forces are conservative! Work done by a conservative force on a particle in a closed loop is zero!

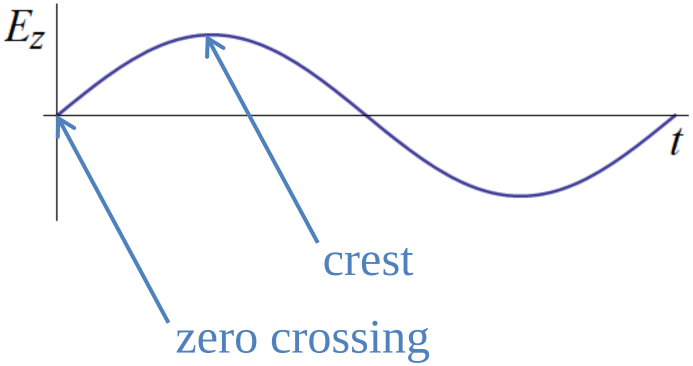


- Solution: Electromagnetic waves confined in cavities.
- The frequency range determines the cavity dimension.

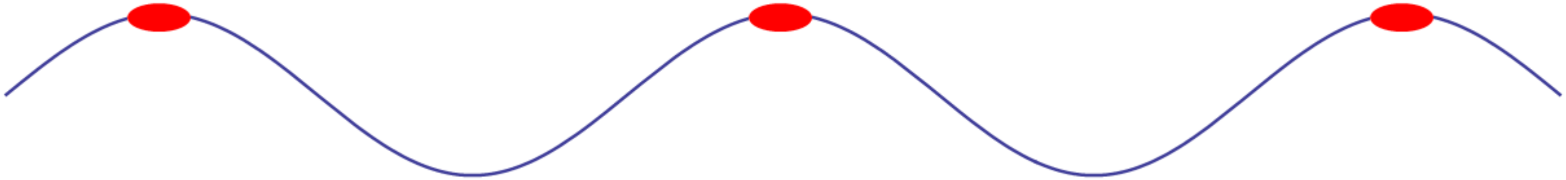
⟹ Radio Frequency (RF) acceleration

RF acceleration and the concept of bunching

- Sinusoidal field: Alternative acceleration and deceleration depending on the arrival time.



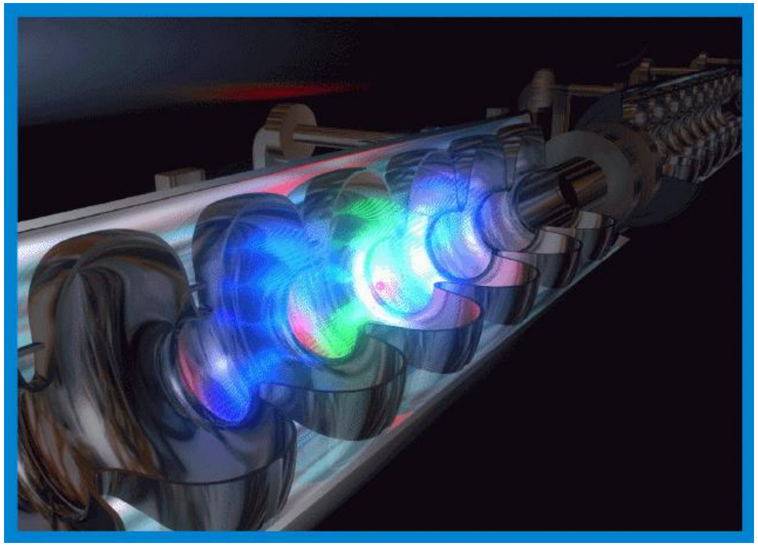
- Bunched beam



3. RF cavities

RF cavities

- An RF cavity is simply an empty space surrounded by metallic walls in which the electromagnetic waves *resonate*.
- In accelerators, one usually uses cavities with *axisymmetric geometries*.



- Cavity modes
  - ✓ Different solutions to Maxwell's equations for any specific boundary condition.
  - ✓ The excited mode depends on the cavity geometry and the frequency.
  - ✓ Transverse electric (TE), transverse magnetic (TM), transverse electric and magnetic (TEM) modes.
  - ✓ transverse magnetic (TM) or E mode is characterized by a longitudinal electric field (E_z).

RF cavities

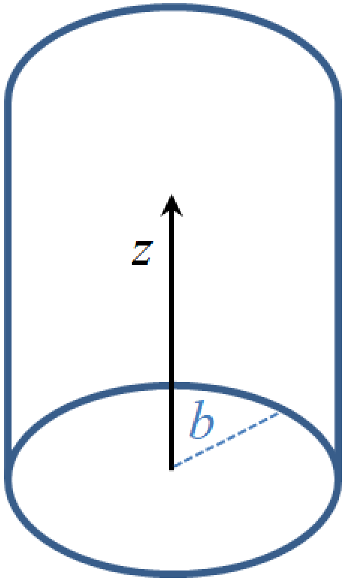
➢ Example:

TM_{01} mode in a cylindrical waveguide [1] (the lowest frequency mode).

$$E_z = E_0 J_0(k_c r) e^{-ik_z z} e^{i\omega t}$$

$$\begin{cases} k = \omega_c / c \\ k_c = \omega_c / c = p_{01}/b \\ k_z^2 = k^2 - k_c^2 \end{cases}$$

first root of $J_0(x)$



$E_z = E_0 \cos\theta$ ⟶ Longitudinal field on axis

$\theta = k_z z - \omega t$ ⟶ RF phase seen by the particle

Synchronization condition: $\frac{d\theta}{dt} = 0 \Longrightarrow v = \boxed{\frac{\omega}{k_z}}$

definition of phase velocity, v_{ph}

$$v_{ph} = c \frac{\omega}{\sqrt{\omega^2 - \omega_c^2}} > c$$

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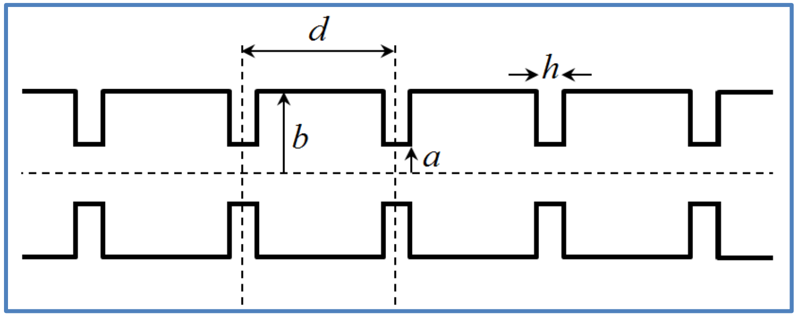
[1] D.M. Pozar, “Microwave Engineering”, third edition, John Wiley & Sons, 2005 (chapter 3).

Disc loaded structure; Slowing down the waves!

- Introducing some obstacles, e.g. irises and providing a periodic structure.
- A perturbed version of the cylindrical waveguide



Periodic function of z

$$E_z = E_d(r,z)e^{-ik_0 z}e^{i\omega t} \qquad r < a$$

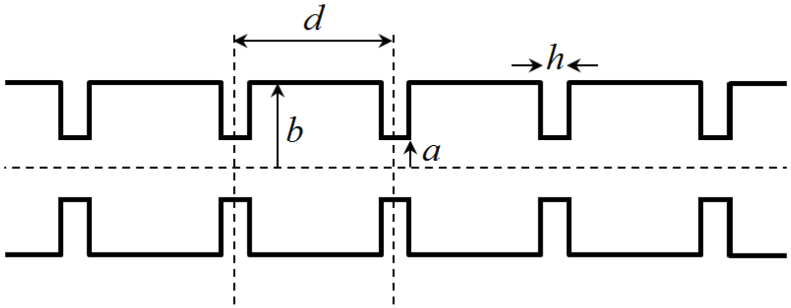
Fourier expansion of $E_d(r,z)$ and applying Maxwell's equations

$$\Longrightarrow \quad E_z(r,z,t) = E_0 \sum_{n=-\infty}^{\infty} C_n J_0(K_n r)e^{i(\omega t - k_n z)} \qquad \begin{cases} k_n = k_0 + 2\pi n/d \\ K_n^2 = (\omega/c)^2 - k_n^2 \end{cases}$$

$$v_{ph,n} = \frac{\omega}{k_n} = \frac{\omega}{k_0 + 2\pi n/d} \Rightarrow$$

Infinite number of waves with different phase velocities (called Space Harmonics).

Structure modes

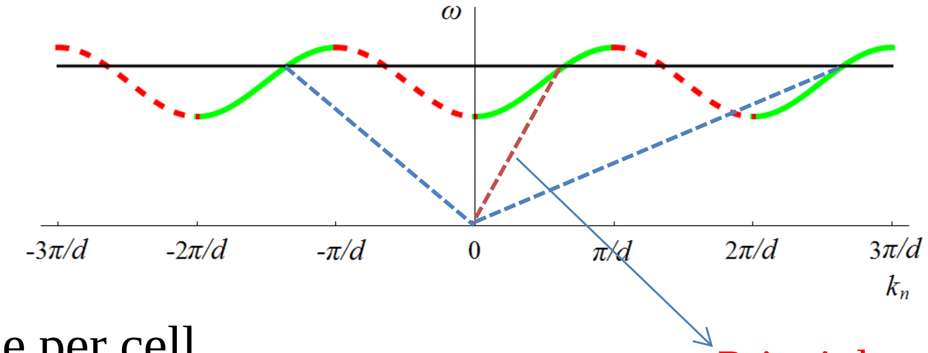


- Slater theorem

  $\Longrightarrow$ dispersion relation [2]:

$$\omega = \frac{p_{01}c}{b}\sqrt{1 + \kappa(1 - \cos k_n d)e^{-\alpha h}}$$

$$\begin{cases} \kappa = \dfrac{4a^3}{3\pi J_1^2(p_{01})b^2 d} \ll 1 \\ \alpha = \sqrt{\left(\dfrac{p_{01}}{a}\right)^2 - \left(\dfrac{\omega}{c}\right)^2} \end{cases}$$



Principle wave ($n = 0$)

- Phase advance per cell

$$\psi = k_0 d \longrightarrow [0, \pi] \qquad \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \ldots$$

Structure modes

$$v_{ph,0} = \frac{\omega}{k_0} = \frac{\omega}{\psi} d \longrightarrow$$ **adjustable phase velocity**

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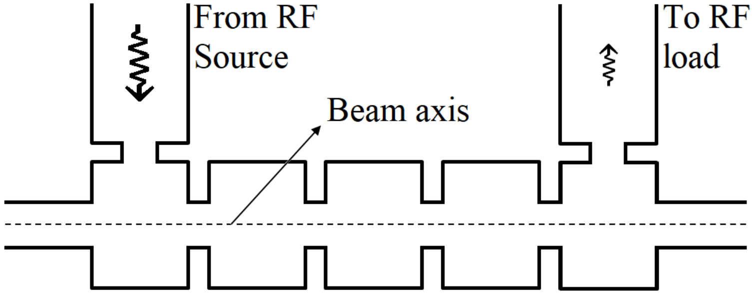
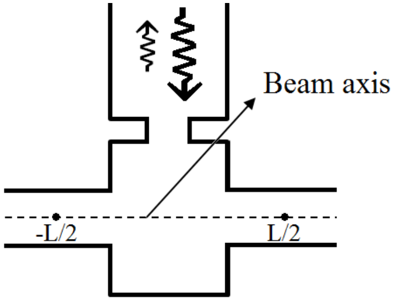
[2] T.P. Wangler et al., “RF Linear Accelerator”, 2nd edition, John Wile & Sons, 2008 (chapter 3).

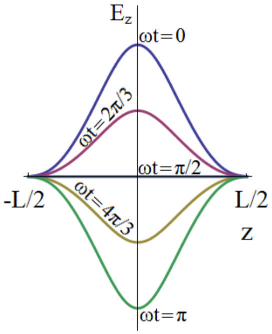
Traveling wave (TW) vs Standing wave (SW) cavities

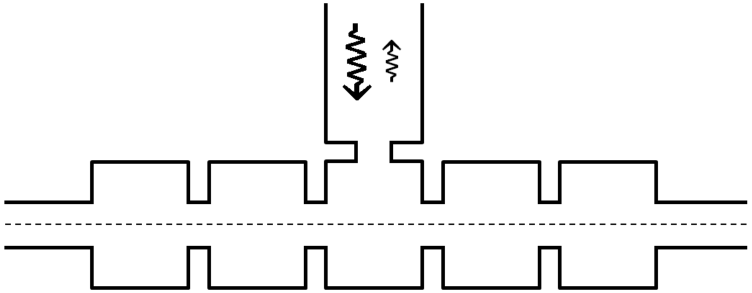
- ➢ TW structure
  - ✓ $E_z^{(TW)}(r,z,t) = \mathcal{E}_z^{(TW)}(r,z)e^{i(kz-\omega t)}$
  - ✓ Short filling time ($< 1\mu s$) .



- ➢ SW structure
  - ✓ $E_z^{(SW)}(r,z,t) = \mathcal{E}_z^{(SW)}(r,z)e^{i\omega t}$ (Fixed nodes)
  - ✓ Longer filling time ($\sim 10s\ \mu s$) .



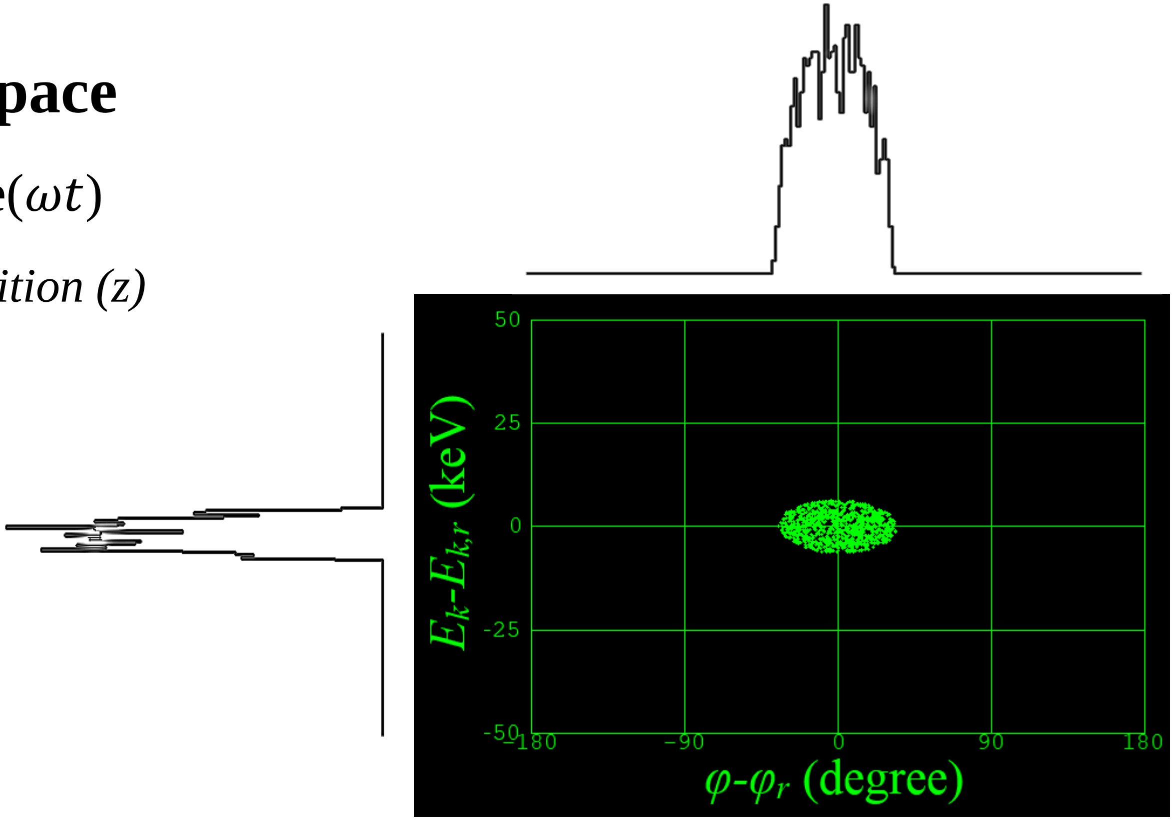




4. Longitudinal beam parameters

Longitudinal phase space

- ➢ Kinetic energy vs phase(ωt)
  - ✓ *At a longitudinal position (z)*
  - ✓ z vs t



- ➢ Beam parameters:
  - ✓ rms bunch length:

$$\sigma_\varphi = \sqrt{\langle(\varphi - \langle\varphi\rangle)^2\rangle}$$

  - ✓ rms energy spread:

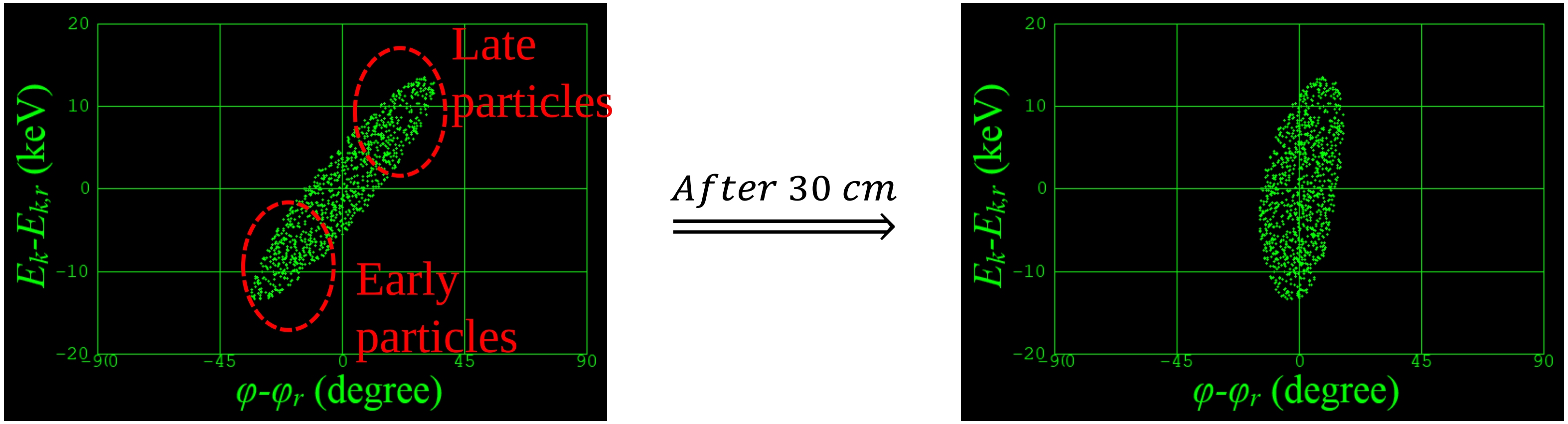
$$\sigma_{E_k} = \sqrt{\langle(E - \langle E\rangle)^2\rangle}$$

  - ✓ rms longitudinal emittance: ⟶ a measure of the area of phase space ellipse.

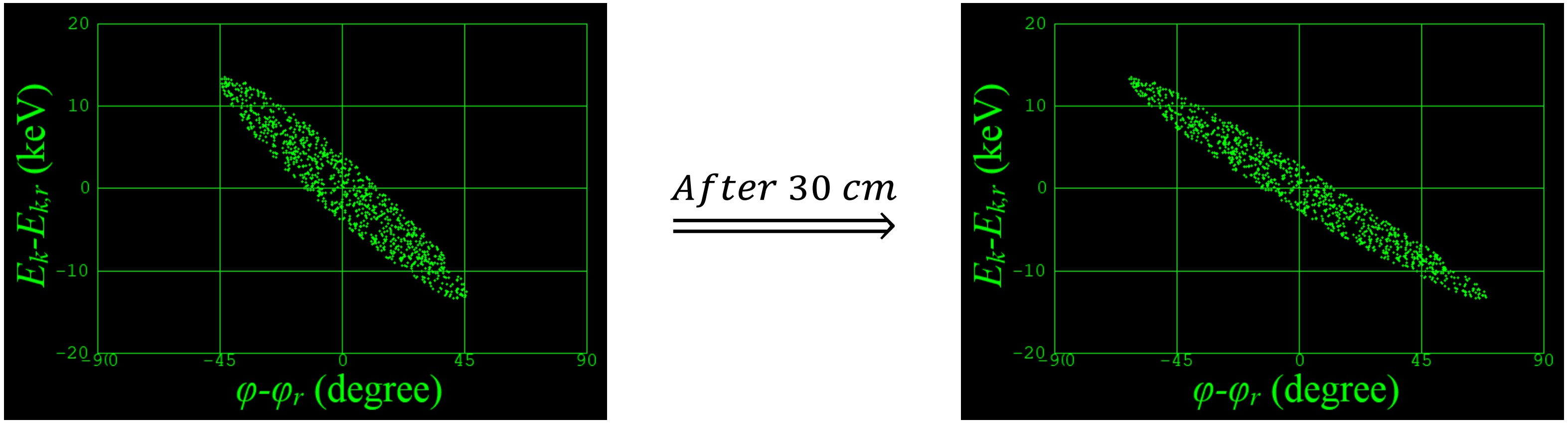
$$\sigma_{E_k} = \sqrt{\langle(E - \langle E\rangle)^2\rangle\langle(\varphi - \langle\varphi\rangle)^2\rangle - \langle(E - \langle E\rangle)(\varphi - \langle\varphi\rangle)\rangle}$$

Longitudinal phase space

- Convergent phase space: $\langle \Delta\varphi \Delta E_k \rangle > 0$



- Divergent phase space: $\langle \Delta\varphi \Delta E_k \rangle < 0$



5. Velocity modulation bunching

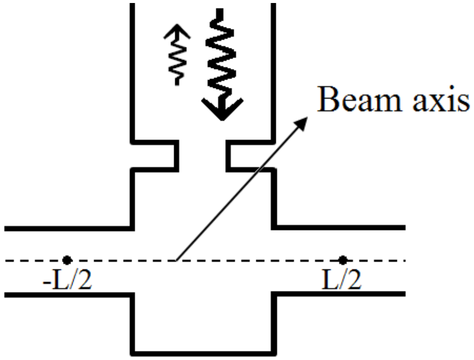
Thin lens cavity and velocity modulation

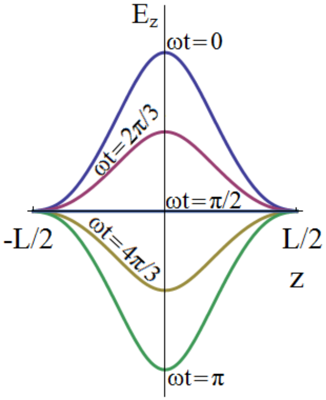
- Energy gain in short cavity

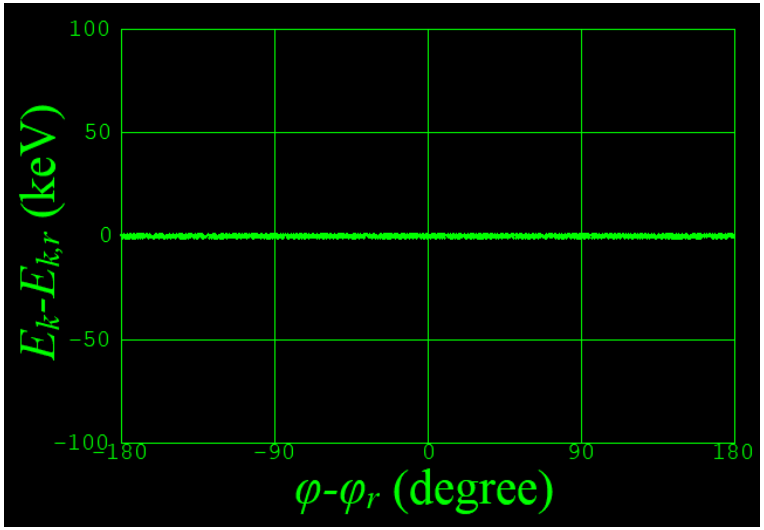
$$\Delta E_k = q\int_{-L/2}^{L/2} Re\left[E_z^{(SW)}(0,z,t)\right] dz$$

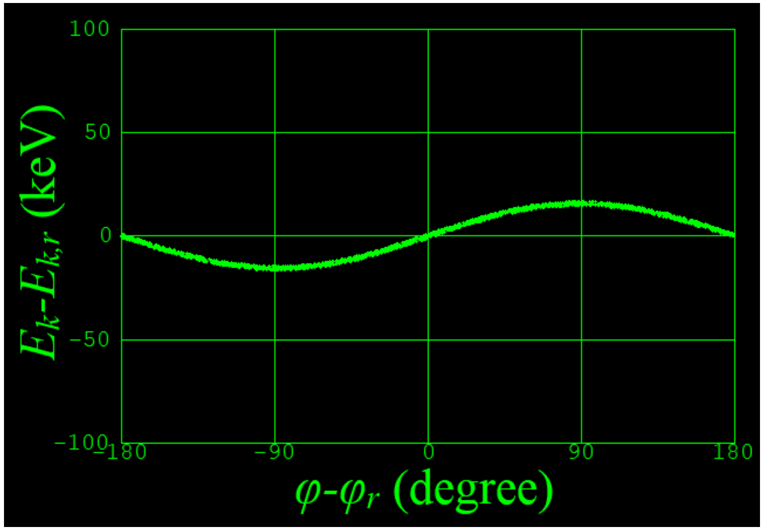
$$= q\int_{-L/2}^{L/2} \mathcal{E}_z^{(SW)}(0,z)\cos\omega t\, dz$$

$$= qV\cos\omega t \longrightarrow$$ Definition of the cavity voltage

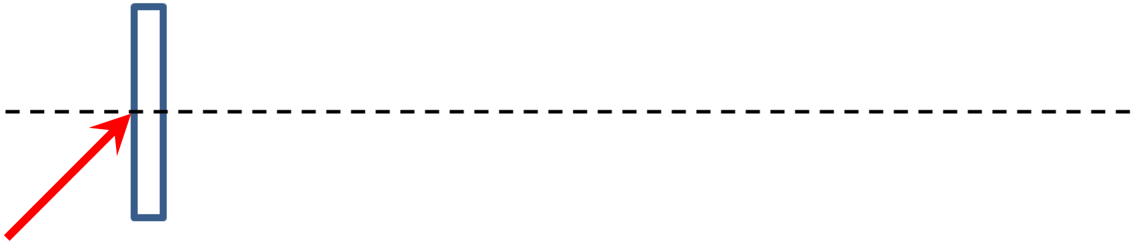


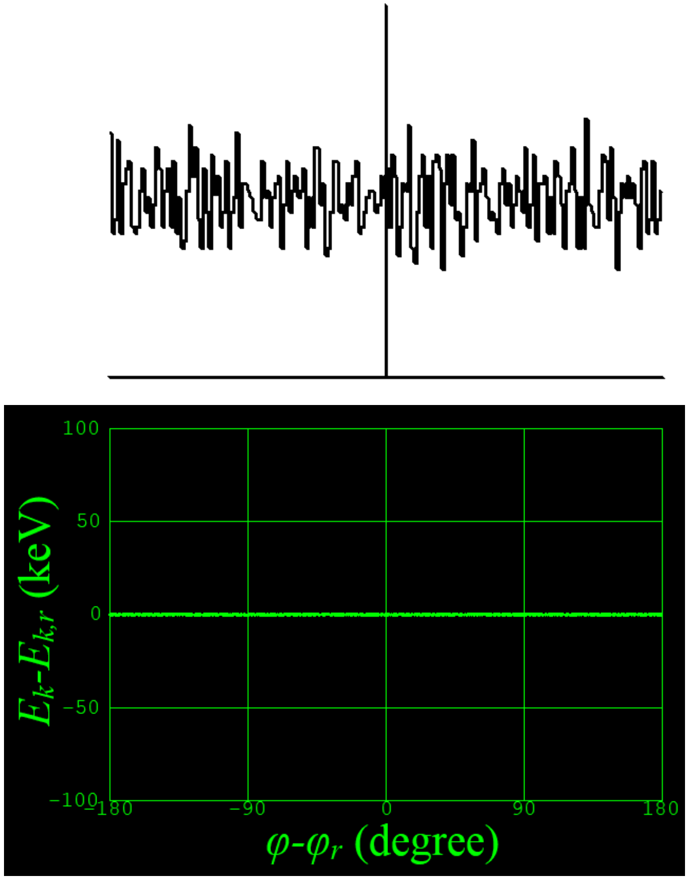






Velocity modulation bunching





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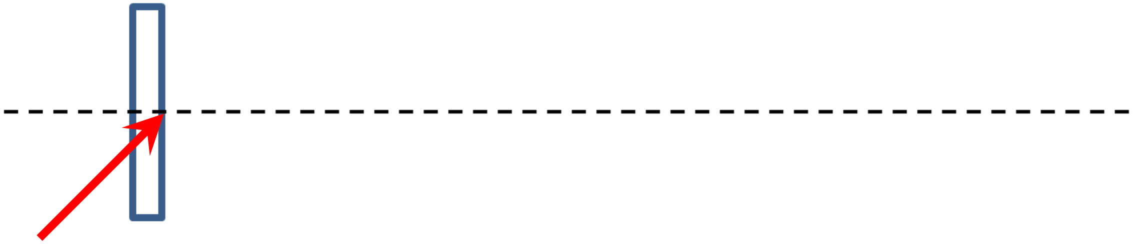

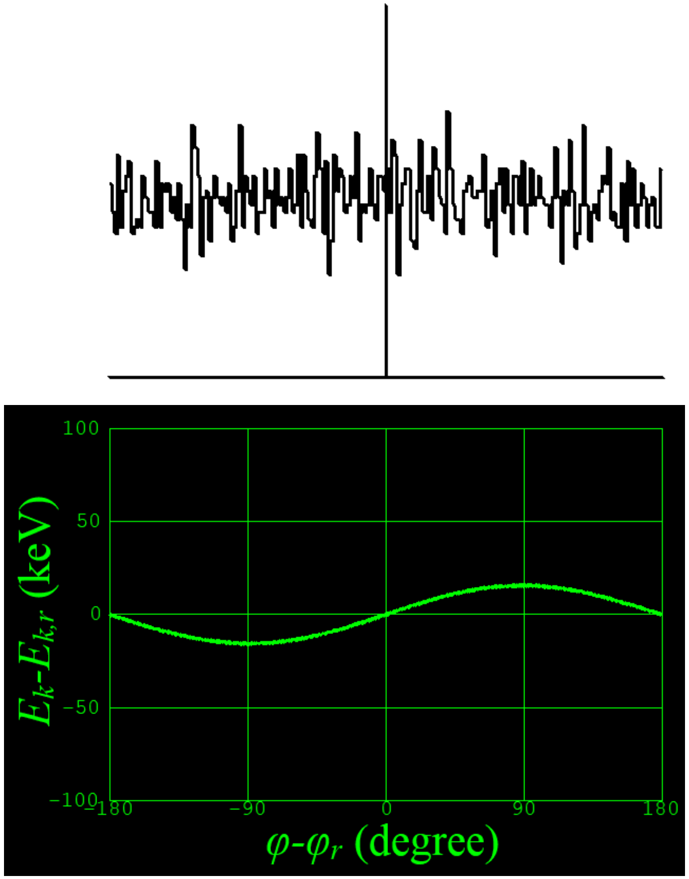

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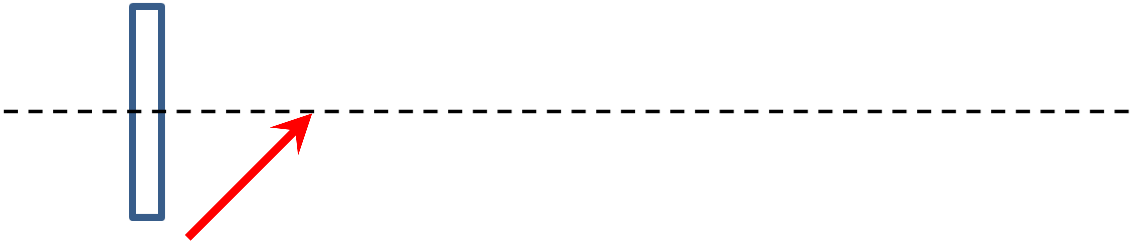

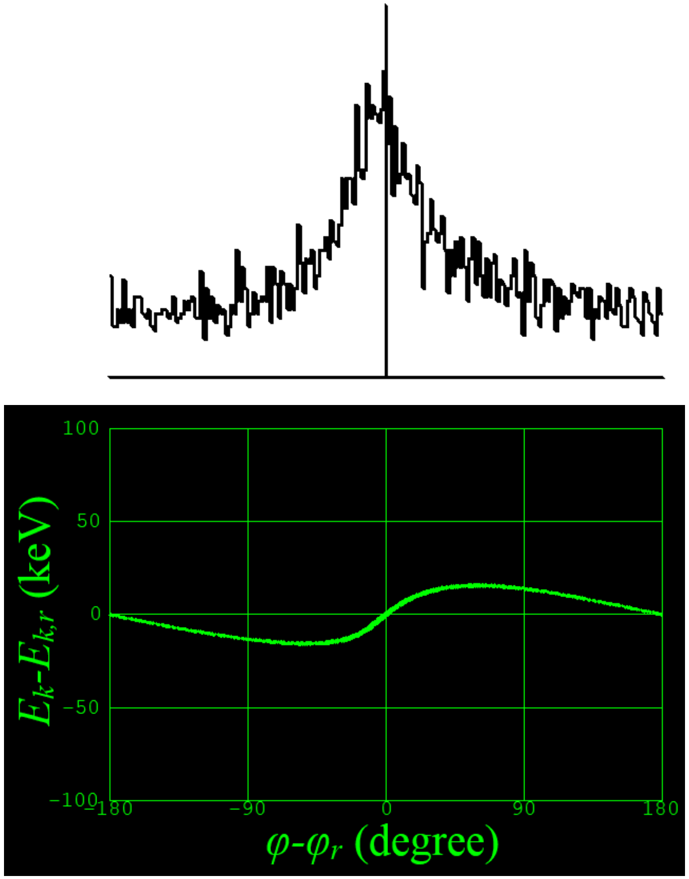

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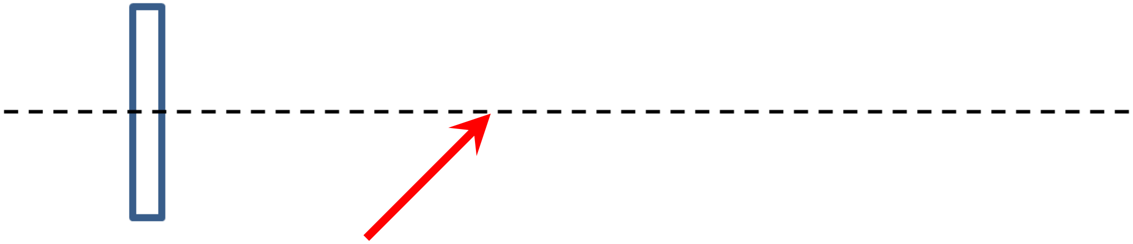

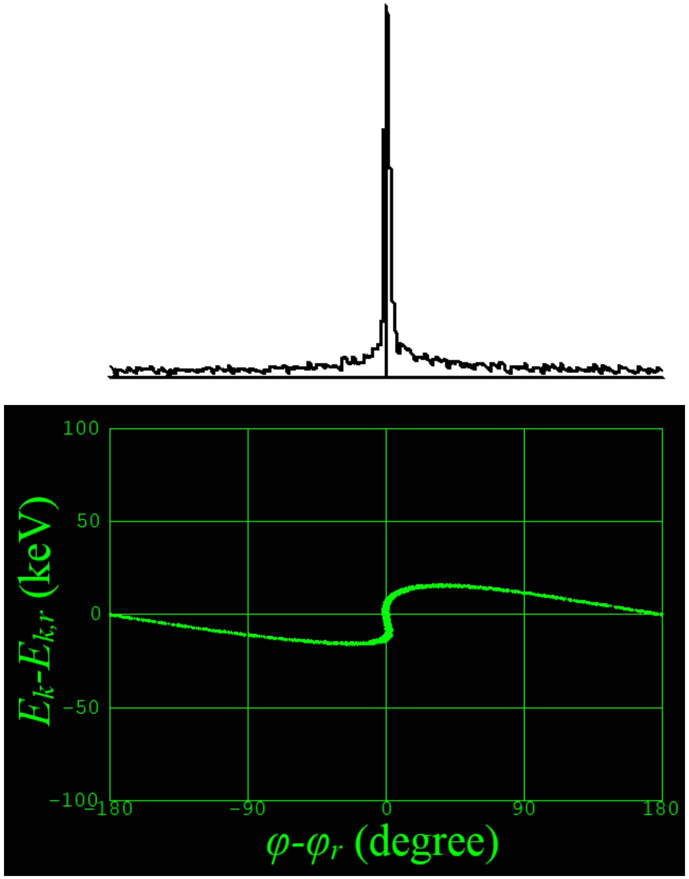

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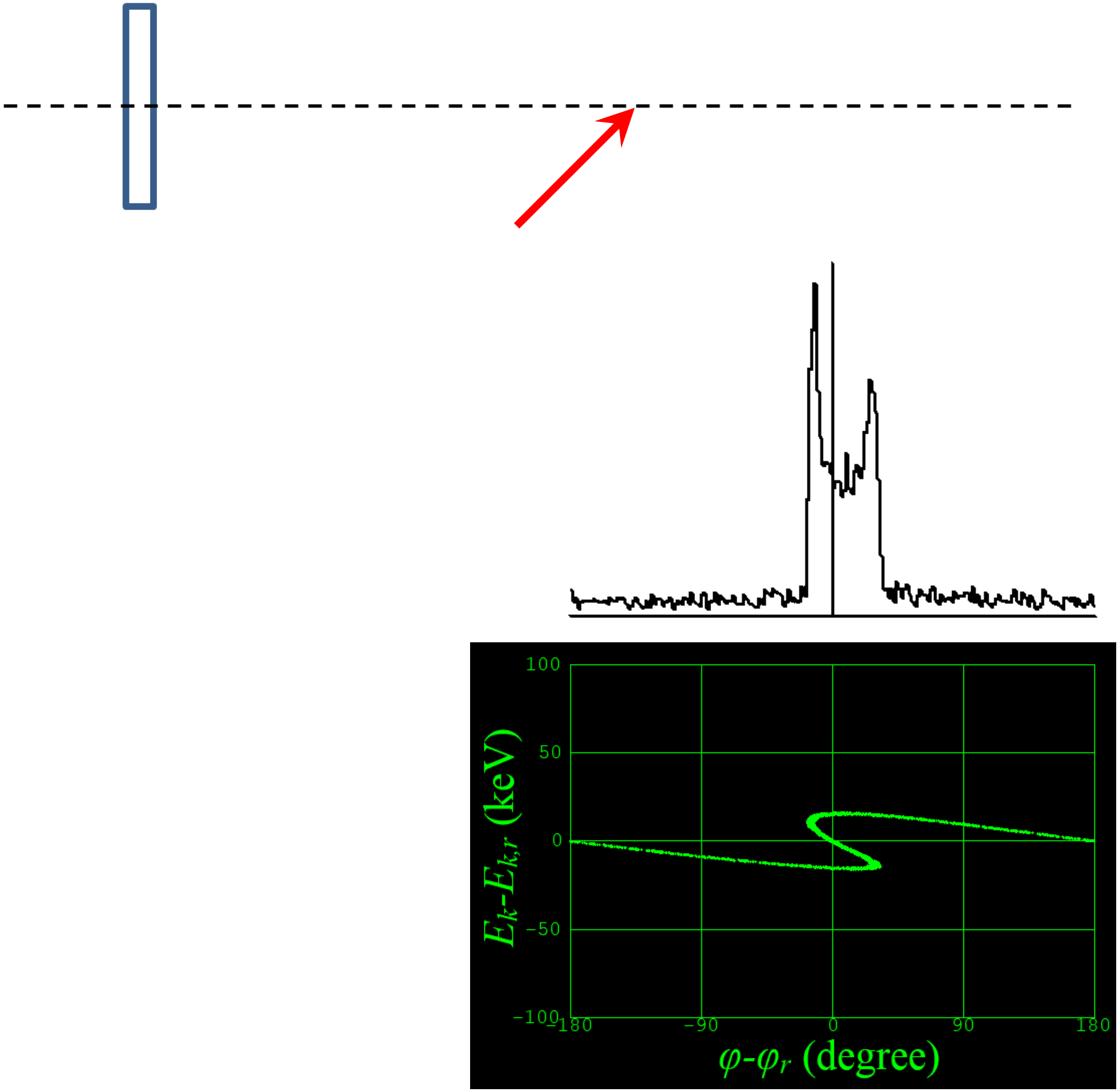

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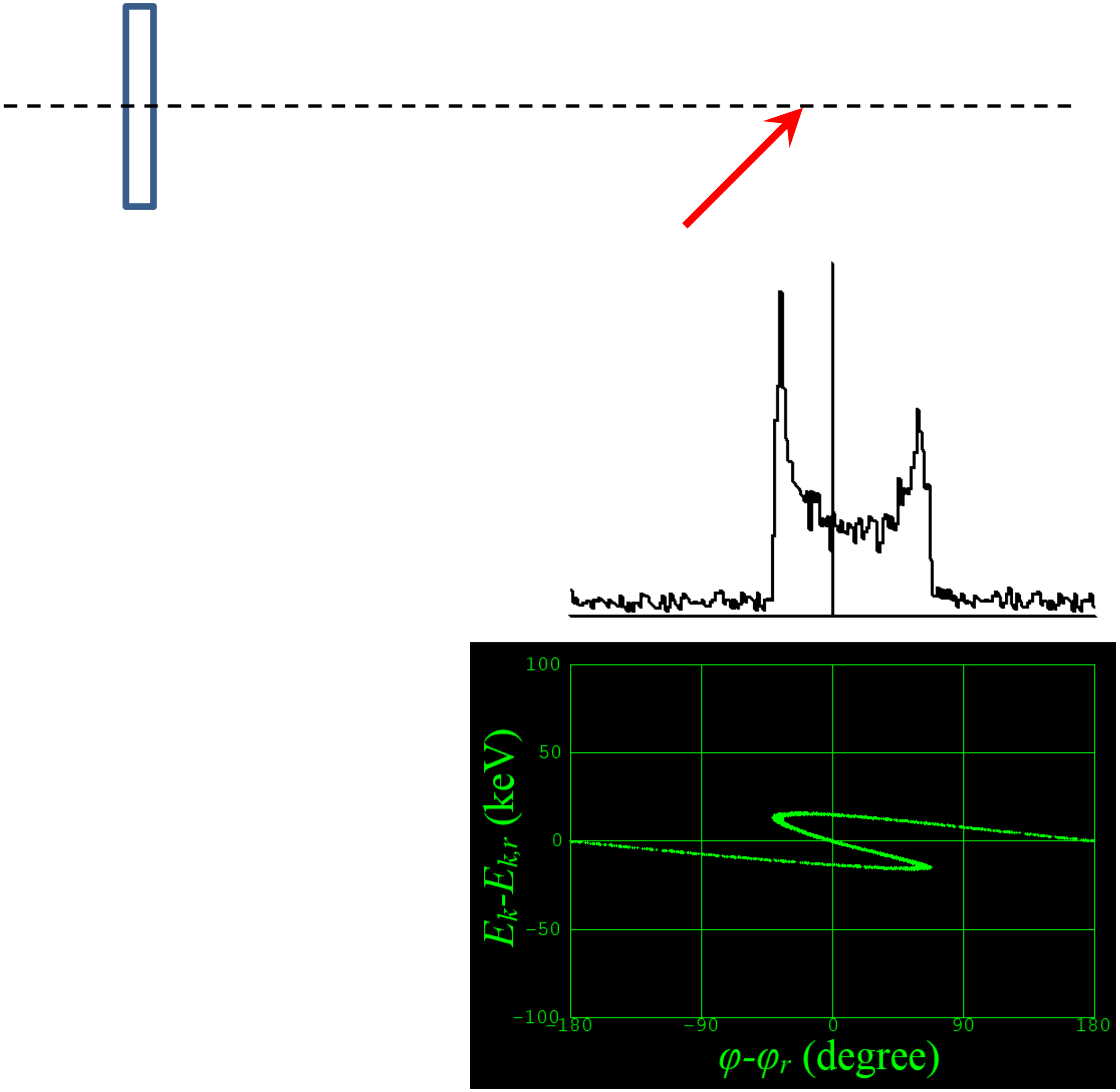

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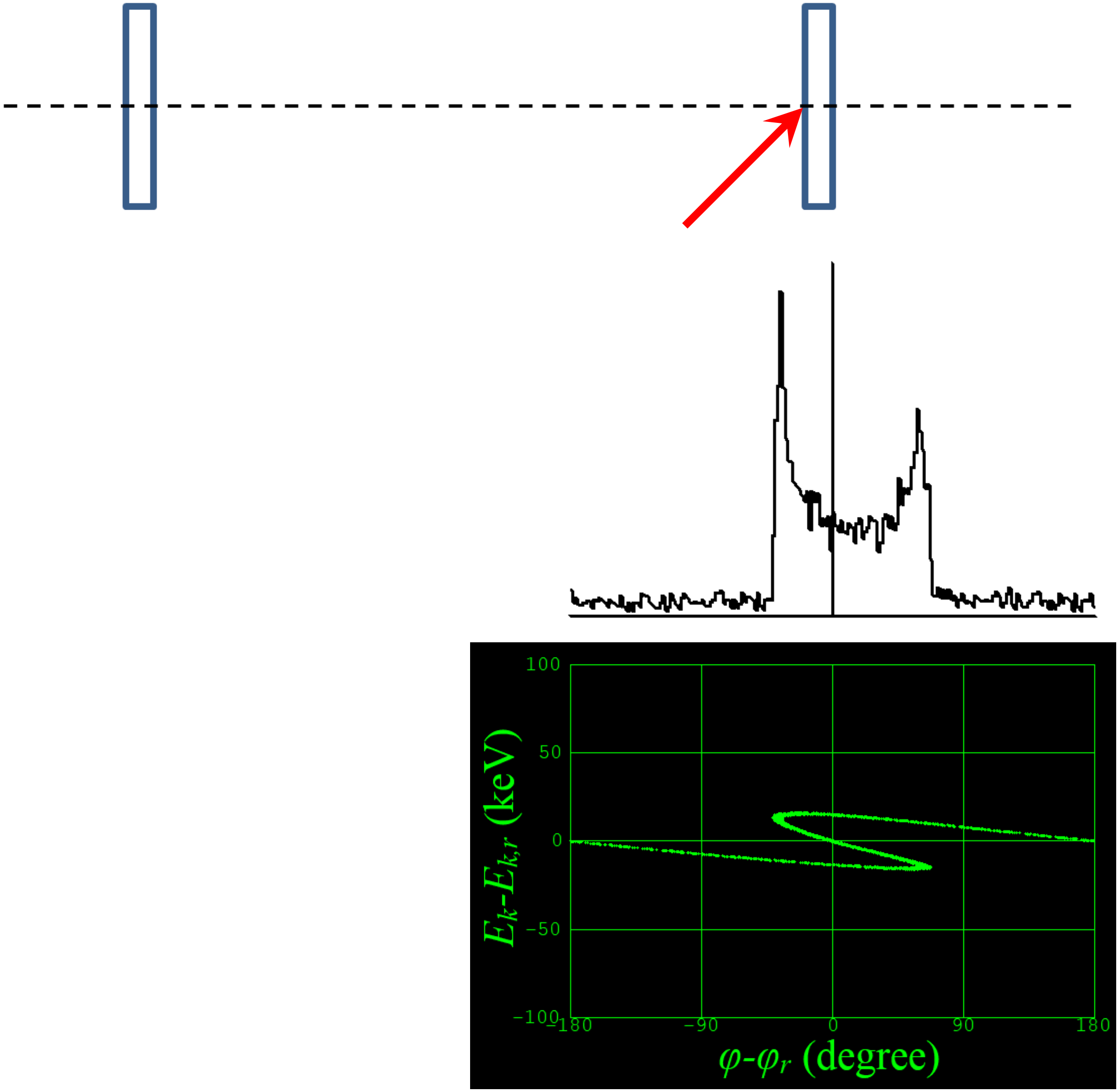

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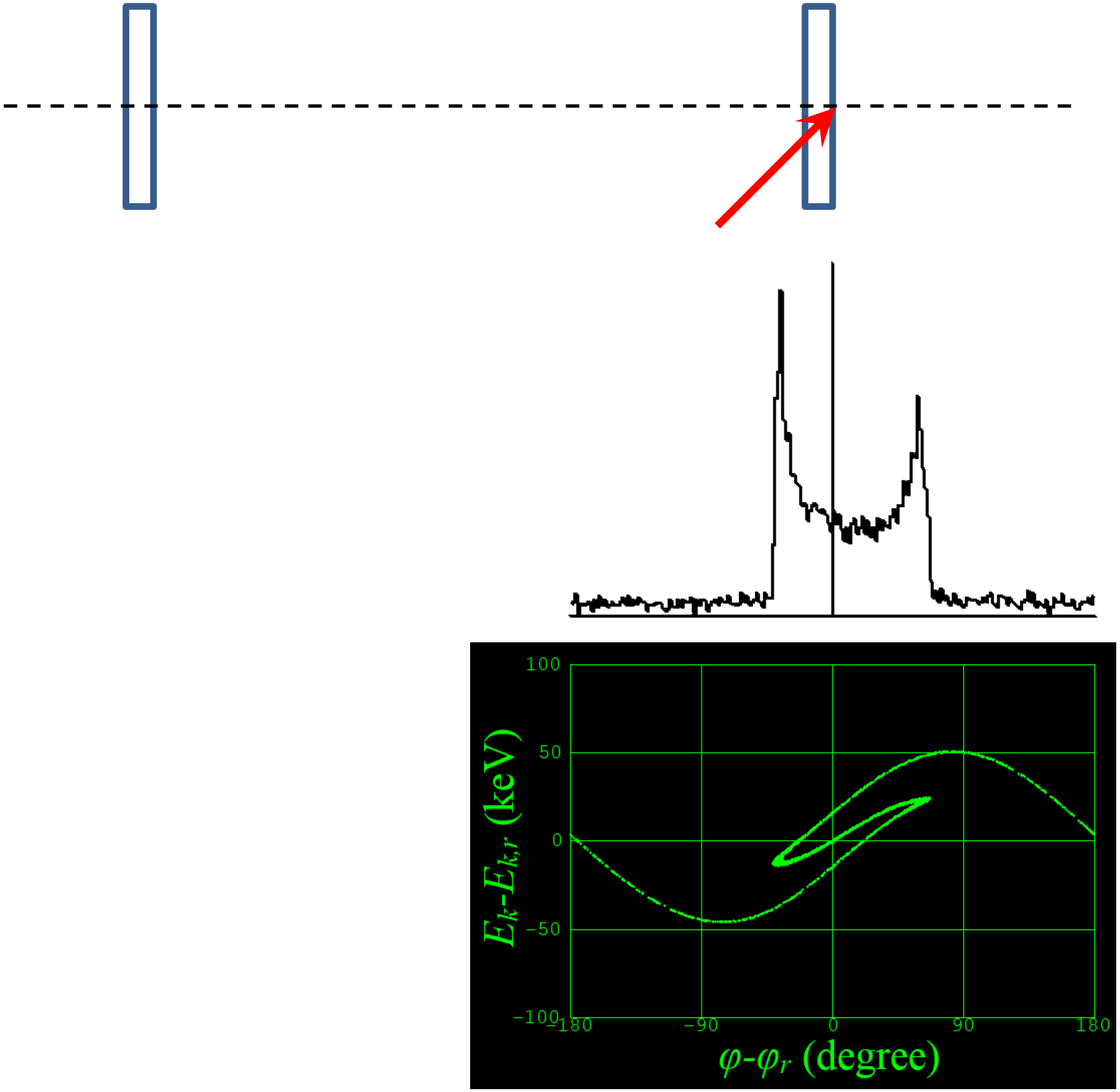

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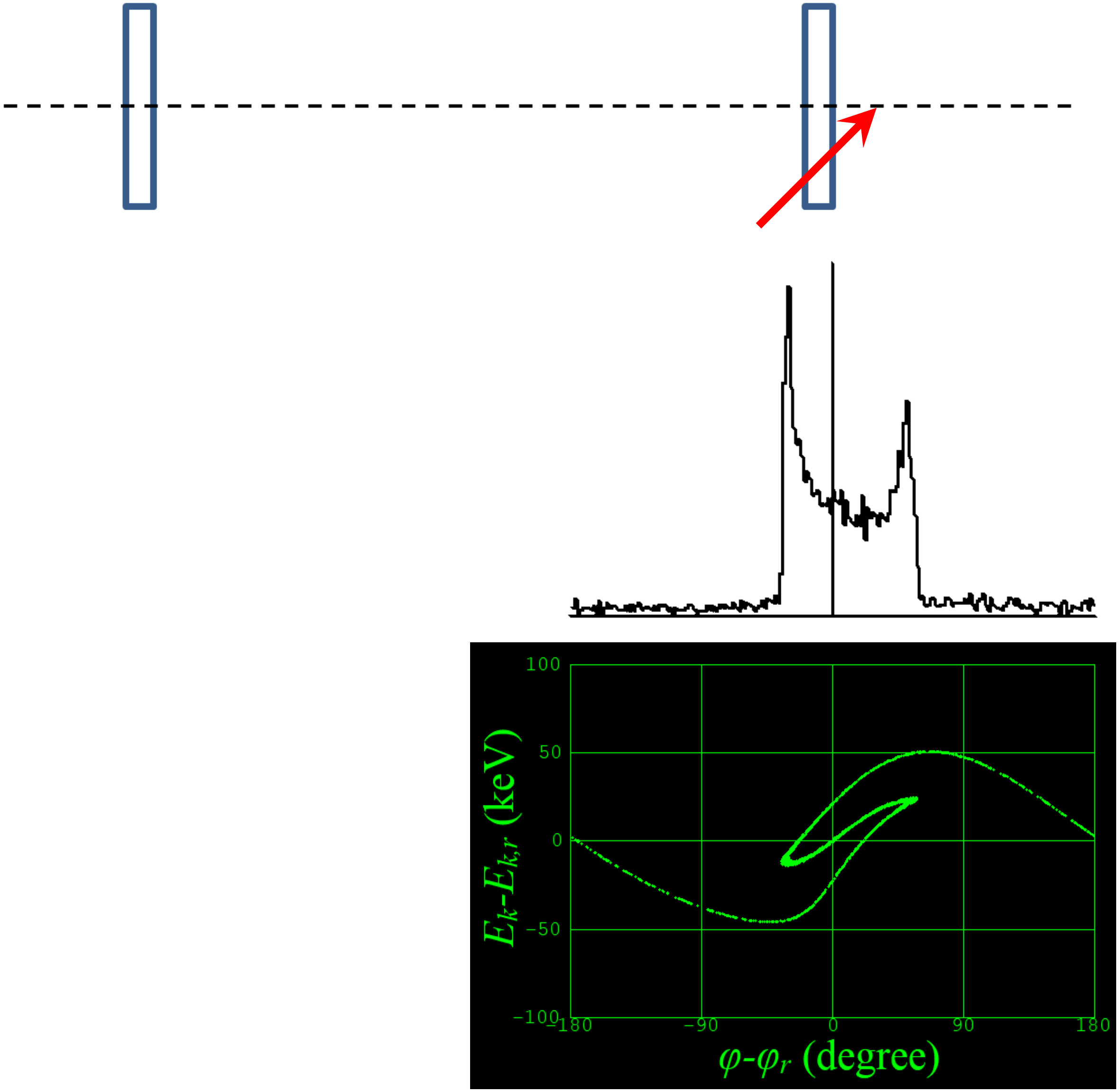

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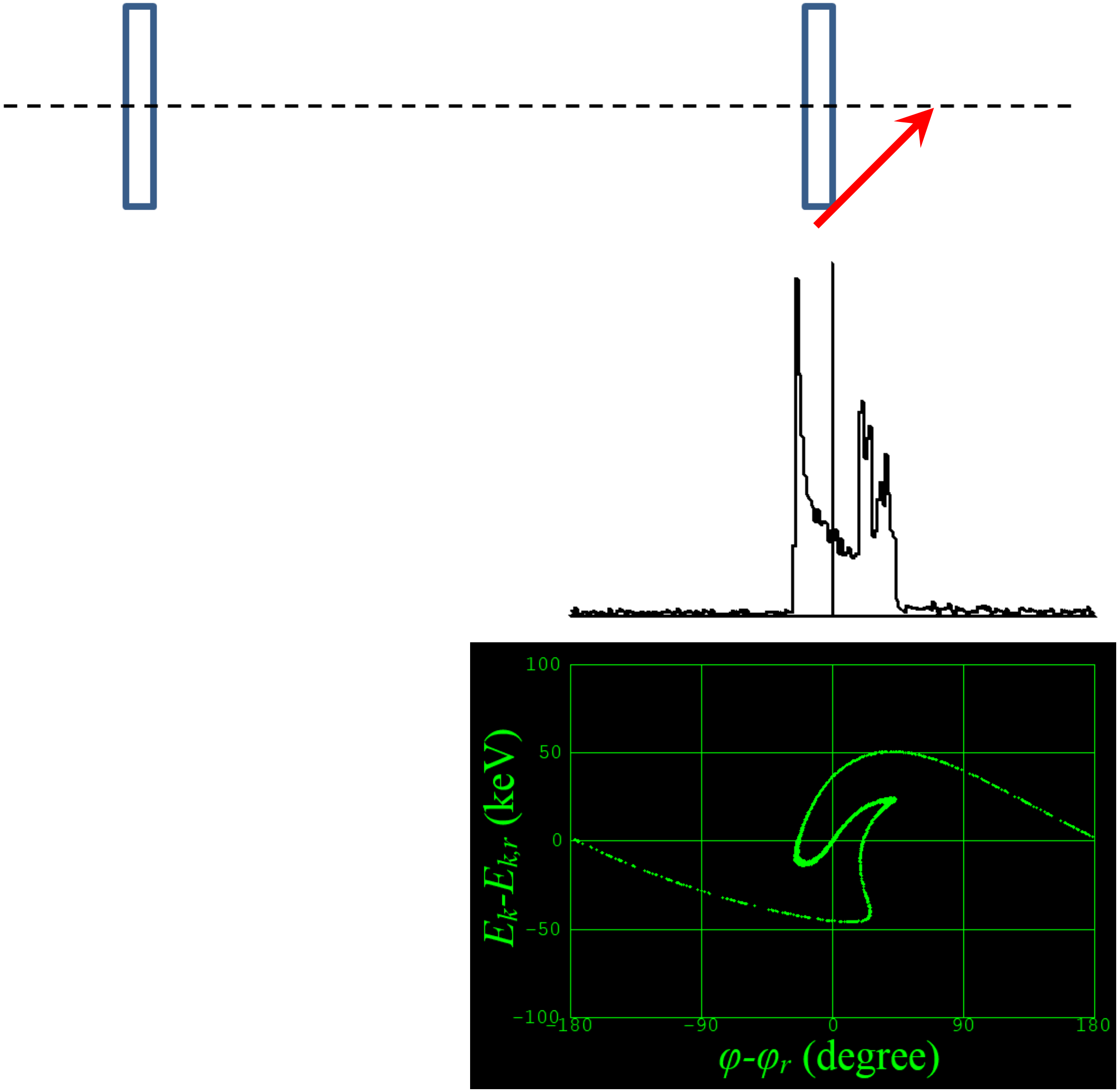


6. Adiabatic phase damping

Longitudinal Beam dynamics in a TW structure

- ➢ Longitudinal electric field
  - ✓ $E_z(r,z,t) = E_0 \sum_{n=-\infty}^{\infty} C_n J_0(K_n r) e^{i(\omega t - k_n z)}$
    1. In most of structures of interest $C_n \ll C_0 \quad , n \neq 0$.
    2. The structure is design so that $v = v_{ph,0}$, therefore the interaction of the particles with other space harmonics ($v \neq v_{ph,n}\, , n \neq 0$) can be neglected.

$$\Longrightarrow \quad Re[E_z(0,z,t)] = E_0 \cos\theta \qquad , \qquad \theta = \omega t - k_0 z$$

- ➢ Longitudinal equation of motions
  - ✓ $\Delta E_k = -eE_0 \cos\theta\, \Delta z = mc^2 \Delta\gamma \quad \Longrightarrow$
  - ✓ $\theta = \omega t - k_0 z = \omega\left(t - \frac{1}{v_{ph}} z\right) \quad \Longrightarrow$

$$\boxed{\begin{aligned} \frac{d\gamma}{dz} &= -\frac{e}{mc^2} E_0 \cos\theta \\ \frac{d\theta}{dz} &= \omega\left(\frac{1}{v} - \frac{1}{v_{ph}}\right) \end{aligned}}$$

- ➢ Definition of synchronous particle
  - ✓ A theoretical particle whose velocity is kept (approximately) equal to v_{ph} derfore $d\theta_s/dz \approx 0$.

Motion of non-synchronous particle

$$\frac{d\gamma}{dz} = -\frac{e}{mc^2} E_0 \cos\theta$$
$$\frac{d\theta}{dz} = \omega\left(\frac{1}{v} - \frac{1}{v_{ph}}\right)$$

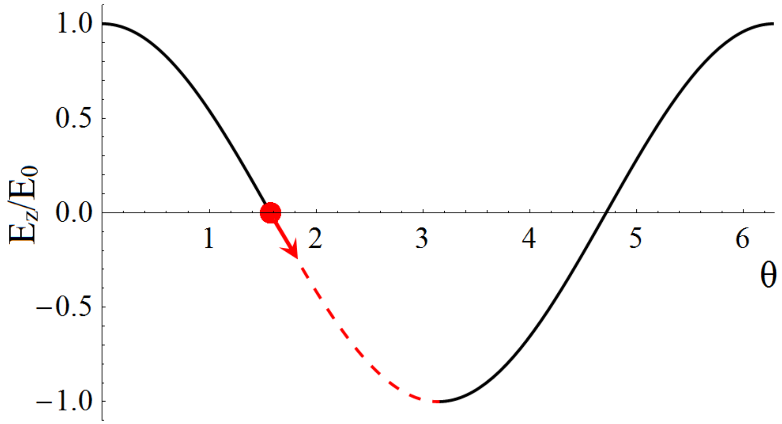
$$\begin{cases} \gamma = \gamma_s + \Delta\gamma \\ \theta = \theta_s + \Delta\theta \end{cases} \xrightarrow{Equation\ of\ motion} \begin{cases} \dfrac{d\Delta\gamma}{dz} = \dfrac{eE_0}{mc^2}\sin\theta_s\,\Delta\theta \\ \dfrac{d\Delta\theta}{dz} = -\dfrac{\omega}{c}(\gamma_s^2 - 1)^{-3/2}\Delta\gamma \end{cases}$$

$$\frac{d^2}{dz^2}\Delta\theta + 2\alpha\frac{d}{dz}\Delta\theta + \Omega^2\Delta\theta = 0 \qquad \text{with} \qquad \begin{cases} \Omega^2 = \dfrac{\omega}{c}(\gamma_s^2 - 1)^{-3/2}\dfrac{e}{mc^2}E_0 \sin\theta_s \\ \alpha = -\dfrac{3\gamma_s}{2(\gamma_s^2 - 1)}\dfrac{e}{mc^2}E_0 \cos\theta_s \end{cases}$$

Beam acceleration ⟹ **Damped oscillatory motion**

The acceleration procedure

- At the entrance of the structure: $v_{ph} = v_s$ and $\cos\theta_s = 0$ or $\theta_s = \pi/2$
- Then v_{ph} is slowly increased. $\Longrightarrow v_{ph} > v_s \;\Longrightarrow \dfrac{d\theta_s}{dz} > 0$



Motion of non-synchronous particle

$$\begin{aligned}\frac{d\gamma}{dz} &= -\frac{e}{mc^2}E_0\cos\theta \\ \frac{d\theta}{dz} &= \omega\left(\frac{1}{v}-\frac{1}{v_{ph}}\right)\end{aligned}$$

$$\begin{cases}\gamma = \gamma_s + \Delta\gamma \\ \theta = \theta_s + \Delta\theta\end{cases} \xRightarrow{Equation\ of\ motion} \begin{cases}\dfrac{d\Delta\gamma}{dz} = \dfrac{eE_0}{mc^2}\sin\theta_s\,\Delta\theta \\ \dfrac{d\Delta\theta}{dz} = -\dfrac{\omega}{c}(\gamma_s^2-1)^{-3/2}\Delta\gamma\end{cases}$$

With a Hamiltonian dynamics approach one finds [3]:

$$\Delta\theta_{max} = C\left[\frac{emc^3}{\omega}\sin\theta_s\,E_0(\gamma_s^2-1)^{3/2}\right]^{-1/4}$$

$$\Delta\gamma_{max} = C\left[\frac{emc^3}{\omega}\sin\theta_s\,E_0(\gamma_s^2-1)^{3/2}\right]^{1/4}$$

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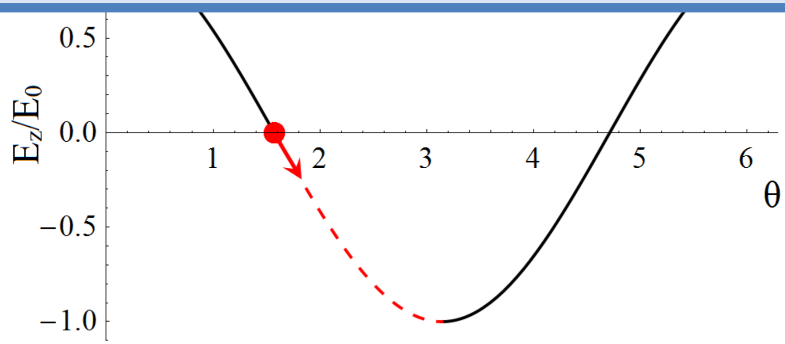
[3] J. Le Duff, Dynamics and Acceleration in Linear Structures, CERN Accelerator School (CAS), CERN-2005-004.

- At the entrance of the structure:
  $v_{ph} = v_s$ and $\cos\theta_s = 0$ or $\theta_s = \pi/2$
- Then v_{ph} is slowly increased.
  $\Longrightarrow v_{ph} > v_s \Longrightarrow \dfrac{d\theta_s}{dz} > 0$

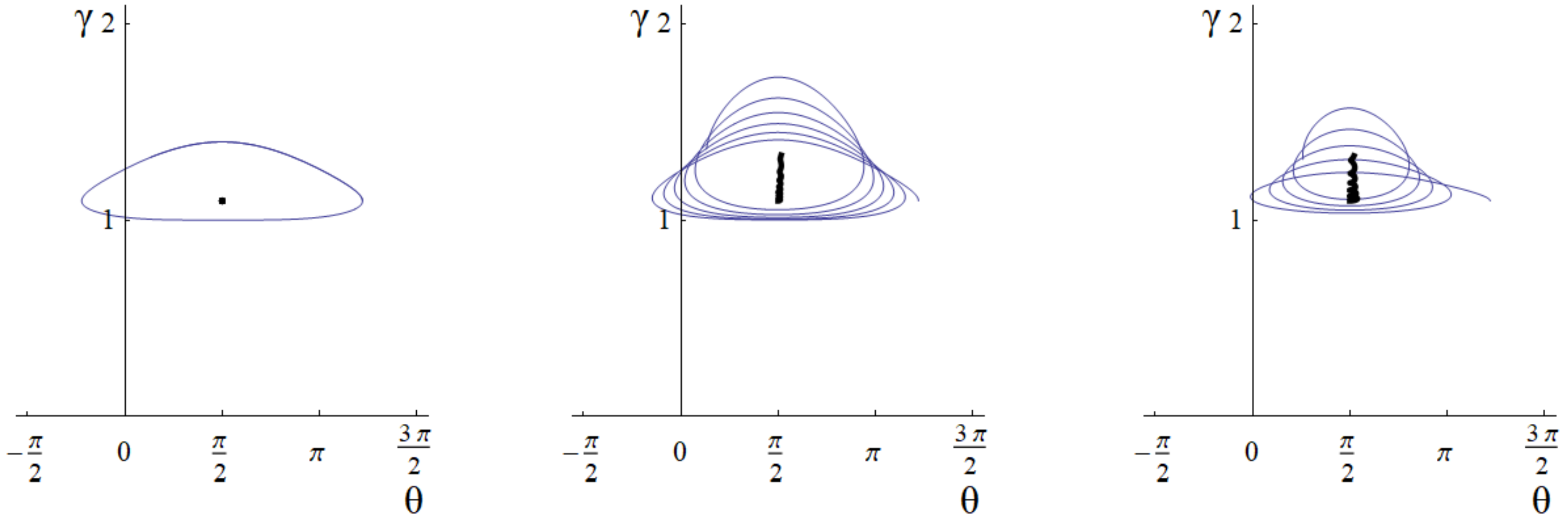


Motion of non-synchronous particle

- Example:

  particle trajectories in three sample TW structure

$$\frac{d\gamma}{dz} = -\frac{e}{mc^2} E_0 \cos\theta$$
$$\frac{d\theta}{dz} = \omega\left(\frac{1}{v} - \frac{1}{v_{ph}}\right)$$



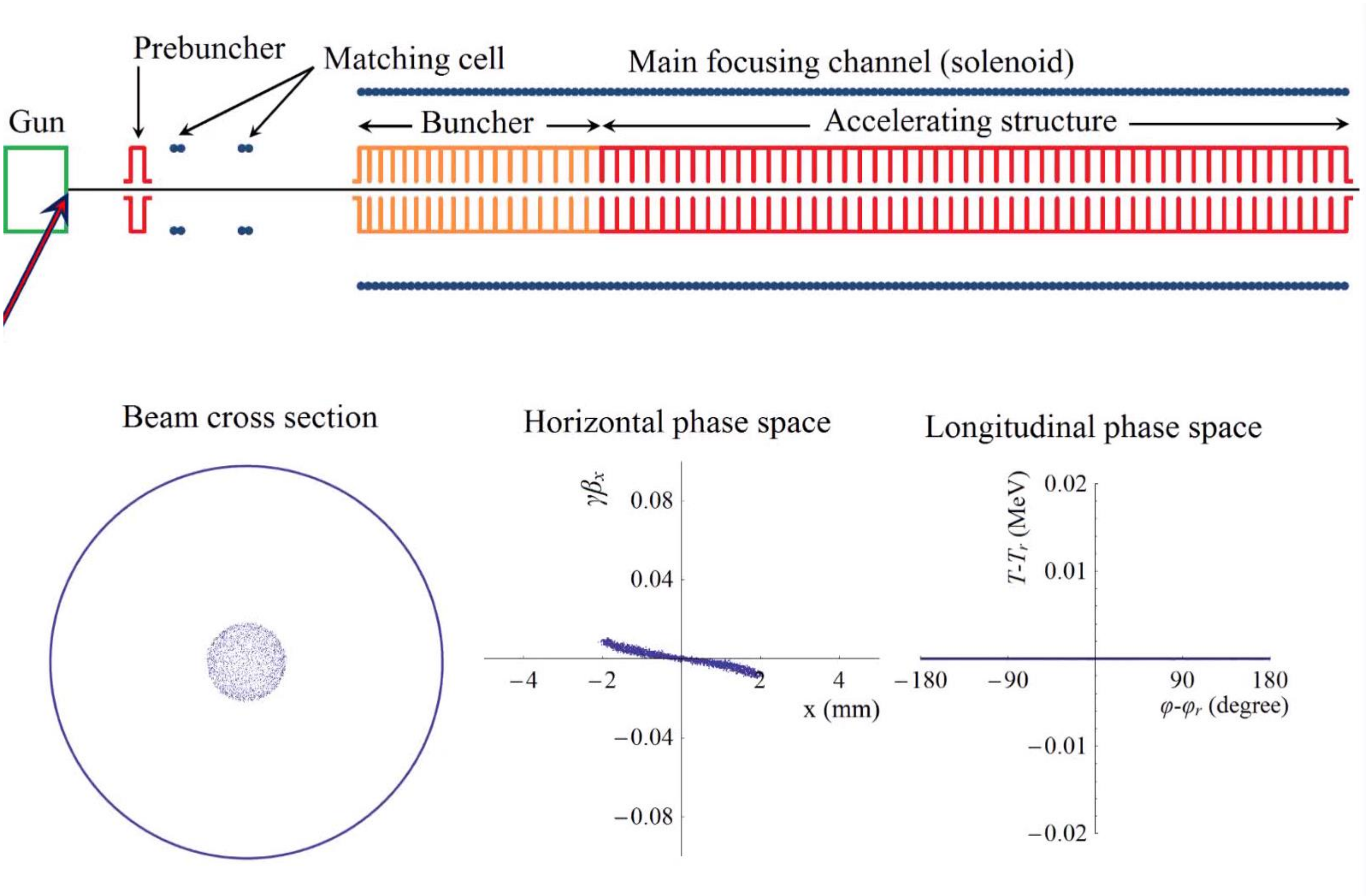
v_{ph} → constant
E_0 → constant

v_{ph} → linearly increased
E_0 → constant

v_{ph} → linearly increased
E_0 → linearly increased

7. Longitudinal dynamics of the IPM linac

8. **Longitudinal dynamics of the IPM linac**



Thanks for your attention!