بسم الله الرحمن الرحیم

Transverse Beam Dynamics

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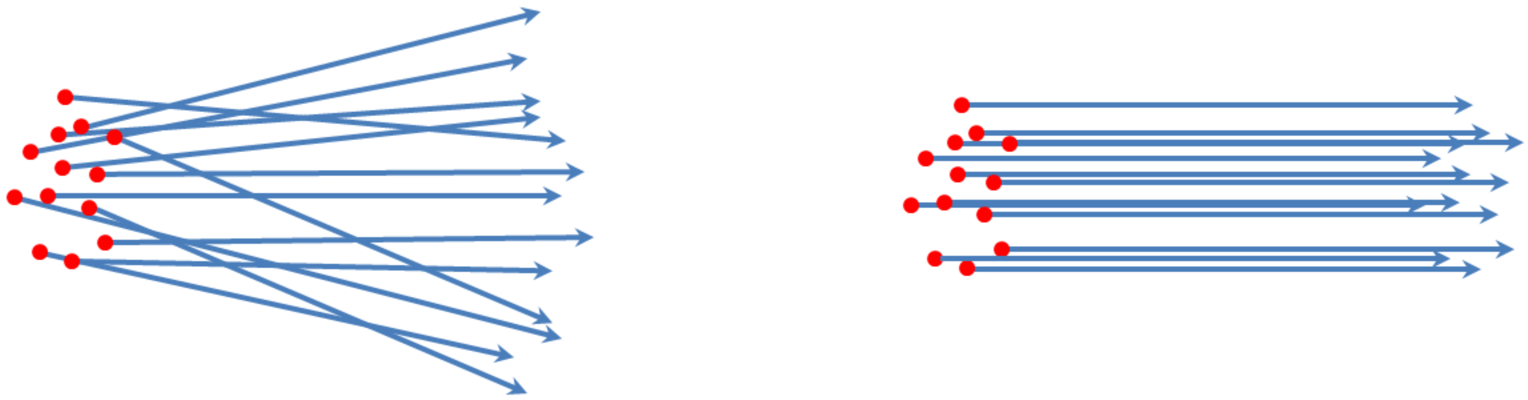
Contents

1. Introduction

The need for focusing

- Space charge repulsive forces.
- Transverse momentum components (the concept of emittance) results in Beam divergence.

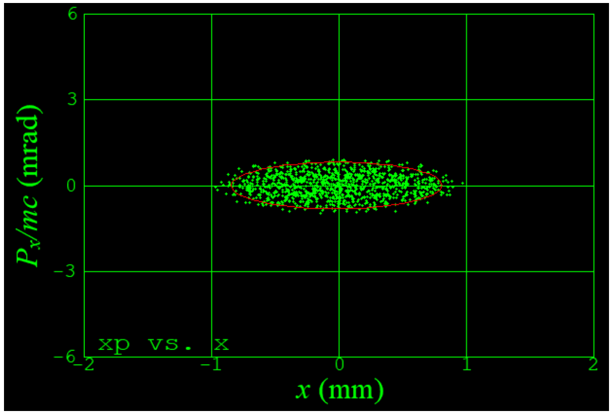


$\Longrightarrow$ Restoring force

- Acceleration vs Beam control
  - ✓ Beam loss
  - ✓ Beam quality

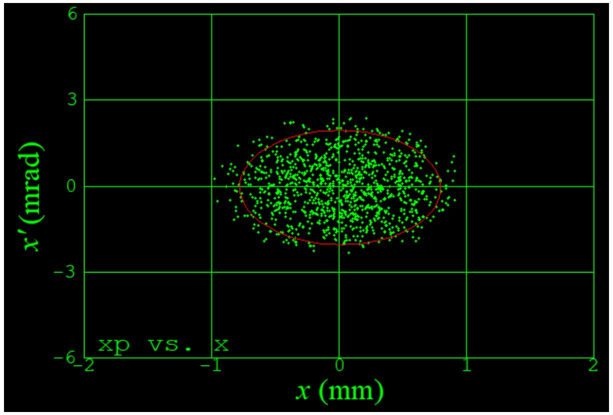
2. Transverse Beam parameters

Transverse phase space



- momentum vs position
- Liouville's theorem
  - ✓ Under conservative forces the area of the phase space remains constant.

Trace space



- Angle vs position

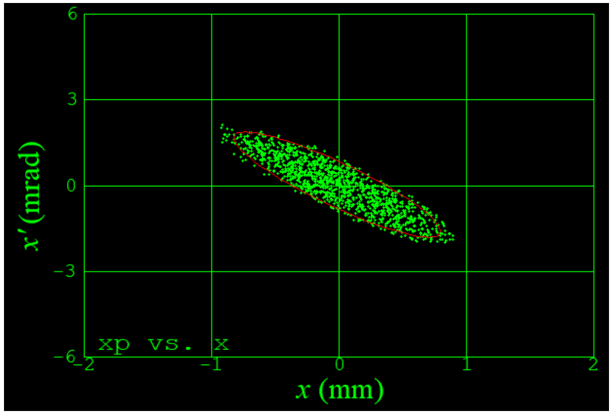
$$p_x = \gamma m \frac{dx}{dt} = \gamma m \frac{dx}{dz} \beta c = mc\gamma\beta x'$$

- Best describes the trajectory of the beam particles.

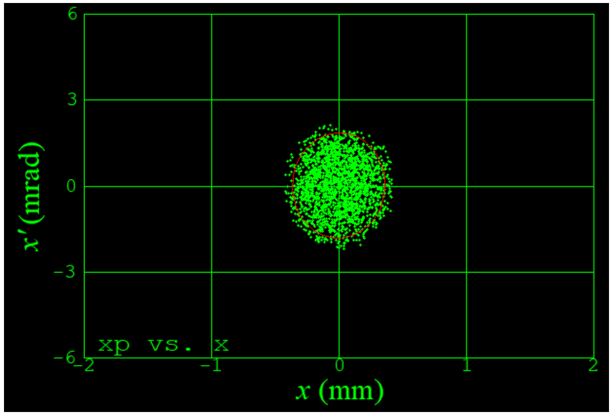
- Adiabatic damping:
  - ✓ $x' \swarrow$ as a result of acceleration
- Measurable quantity

Trace space

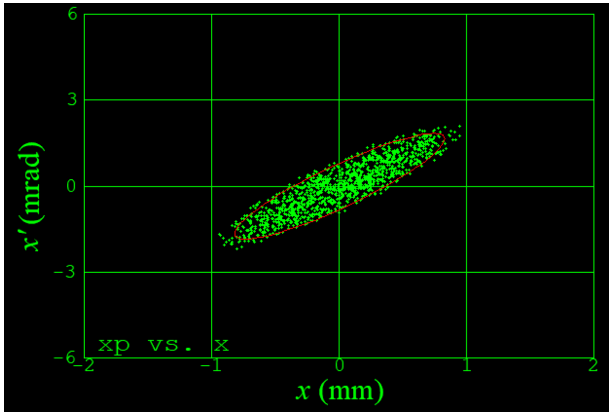
- Convergent beam: $\langle xx' \rangle < 0$



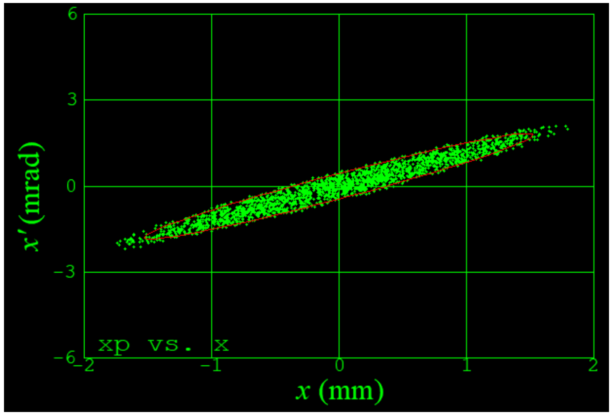
$\underrightarrow{After\ 30\ cm}$



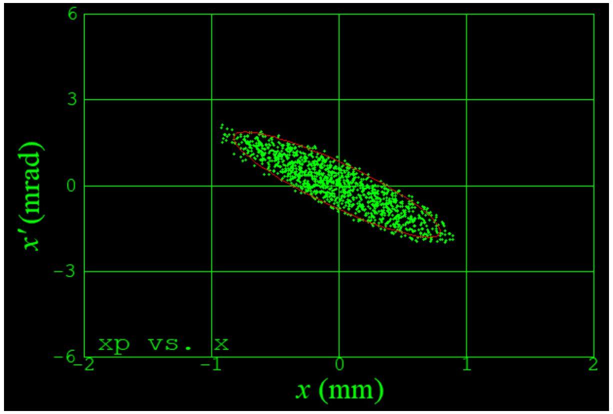
- Divergent beam: $\langle xx' \rangle < 0$



$\underrightarrow{After\ 30\ cm}$



Transverse beam parameters



- Beam position:

  $\langle x \rangle$

- rms Beam size:

  $a = \sqrt{\langle x^2 \rangle}$

- Derivative of the rms Beam size:

  $a' = \langle xx' \rangle / a$

- Geometric emittance: ⟶ a measure of the trace space area.

  $\varepsilon_g = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$

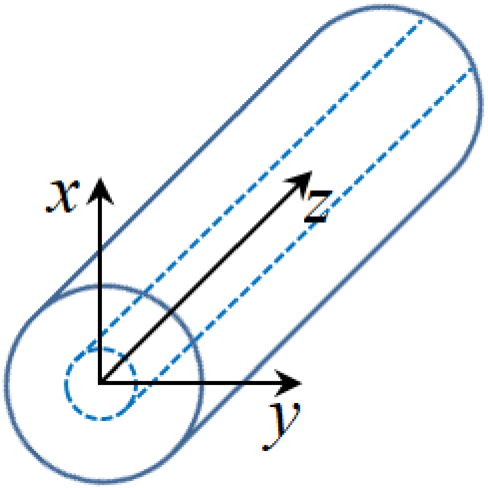
- Normalized emittance: ⟶ a measure of the phase space area.

  $\varepsilon_n = \gamma \beta \varepsilon_g$

3. Focusing

Space charge force

- The force seen by each particle from the charge distribution of the beam.
- Electric field:
  - ✓ Gauss's law: $E_{sr}(r) = \frac{q}{\epsilon_0 r}\int_0^r n(r')r'dr'$
- Magnetic field:
  - ✓ Ampere's law: $B_{s\theta}(r) = \frac{q\mu_0\beta c}{r}\int_0^r n(r')r'dr'$



- Lorentz force
  - ✓ $F_{sr}(r) = \frac{q^2}{\epsilon_0 r}(1-\beta^2)\int_0^r n(r')r'dr'$ ⟶ non-relativistic phenomenon
    - 1 MeV electron beam: $\beta = 0.941$
    - 1MeV proton beam: $\beta = 0.046$
  - ✓ For a uniform distribution: $F_{sr}(r) \propto r$
  - ✓ $E_{sr}, B_{s\theta}(r) \propto I$

External fields

- Electrostatic or magnetostatic lens?
  - ✓ $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$
  - ✓ $cB = 3 \times 10^8 m/s \times 1T = 300\ MV/m!$

    $\Longrightarrow$ For relativistic beams the magnetic field is more efficient for focusing.

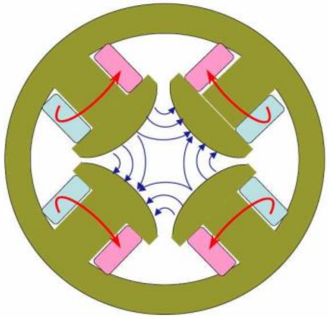
- Longitudinal magnetic field (Solenoids) $\longrightarrow$ later!

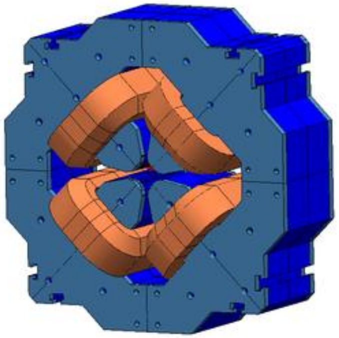
- Transverse magnetic field multipole expansion:

$$B_y = B_{y0} + \left.\frac{\partial B_y}{\partial x}\right|_0 x + \frac{1}{2!}\left.\frac{\partial B_y}{\partial x}\right|_0 x^{\ 2} + \cdots$$

Dipole quadrupole sextupole

- Quadrupole magnet
  - ✓ $\begin{cases} B_y = gx \\ B_x = -gy \end{cases}$





Quadrupole focusing

- Equation of motion

✓ $F_x = \frac{dp_x}{dt} = \gamma m \ddot{x} = e v_z B_y$

$$\begin{cases} \dot{x} = \frac{dx}{dz}\frac{dz}{dt} \cong x'\beta c \\ \ddot{x} \cong x''(\beta c)^2 \end{cases}$$

$$\begin{cases} x'' - kx = 0 \\ y'' + kx = 0 \end{cases}$$

$k = \frac{eg}{\gamma m \beta c} \longrightarrow$ quadrupole strength

$\Longrightarrow$ oscillatory motion around the **orbit**

$$\begin{cases} x = x_0 \cosh\sqrt{k}z + x_0' \frac{1}{\sqrt{k}} \sinh\sqrt{k}z \\ x' = x_0\sqrt{k}\sinh\sqrt{k}z + x_0'\cosh\sqrt{k}z \end{cases}$$

$k > 0 \quad \longrightarrow \quad$ defocusing

$$\begin{cases} x = x_0 + x_0' z \\ x' = x_0' \end{cases}$$

$k = 0 \quad \longrightarrow \quad$ drift

$$\begin{cases} x = x_0 \cos\sqrt{-k}z + x_0' \frac{1}{\sqrt{-k}} \sin\sqrt{-k}z \\ x' = -x_0\sqrt{-k}\sin\sqrt{k}z + x_0'\cos\sqrt{-k}z \end{cases}$$

$k < 0 \quad \longrightarrow \quad$ focusing

Transfer matrix

- Definition

$$\begin{bmatrix} x \\ x' \end{bmatrix} = M \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

- Focusing quadrupole ($k < 0$):

$$M = \begin{bmatrix} \cos\sqrt{-k}l & \frac{1}{\sqrt{-k}}\sin\sqrt{-k}l \\ -\sqrt{-k}\sin\sqrt{-k}l & \cos\sqrt{-k}l \end{bmatrix}$$

- Drift space ($k = 0$):

$$M = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}$$

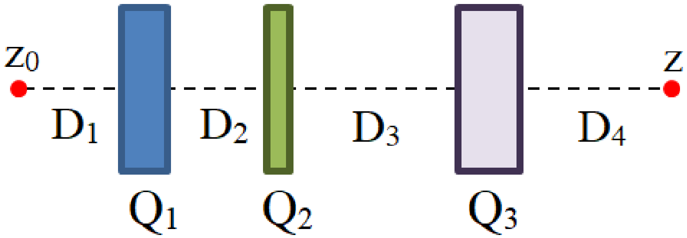
- Focusing quadrupole ($k > 0$):

$$M = \begin{bmatrix} \cosh\sqrt{k}l & \frac{1}{\sqrt{k}}\sinh\sqrt{k}l \\ \sqrt{k}\sinh\sqrt{k}l & \cosh\sqrt{k}l \end{bmatrix}$$

Transfer matrix

- ➢ Multielement transfer line

$$M = M_{D1}M_{Q1}M_{D2}M_{Q2}M_{D3}M_{Q3}M_{D4}$$



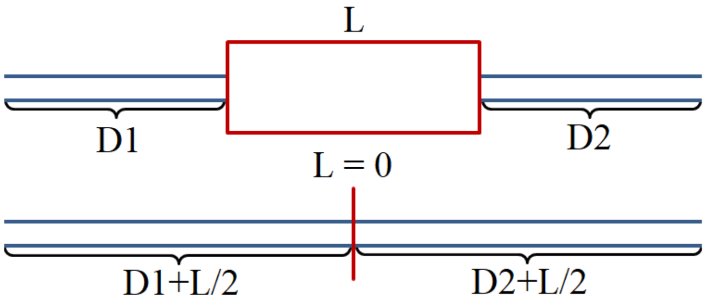
- ➢ Thin lens approximation
  - ✓ $L \to 0$ and $kL = 1/f \neq 0$
  - ✓ Focusing

$$M = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

  - ✓ Defocusing

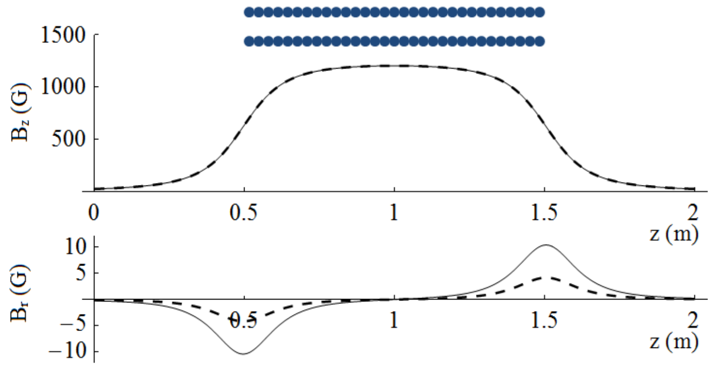
$$M = \begin{bmatrix} 1 & 0 \\ 1/f & 1 \end{bmatrix}$$

Focal length



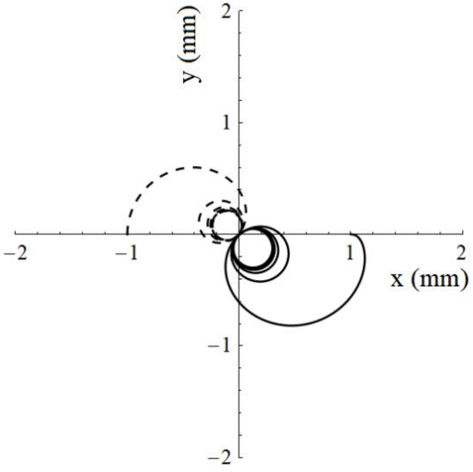
Solenoid focusing

- Field map



- Focusing scheme:
  1. v_z interacts with B_r at the entrance of the solenoid producing a v_φ.
  2. The resulting v_φ interacts with B_z producing F_r.

- Particle trajectories



Solenoid focusing

- Equation of motion

$$\begin{cases} x'' + \frac{q\,B}{\gamma m\beta c} y' + \frac{q\,B'}{2\gamma m\beta c} y = 0 \\ y'' - \frac{q\,B}{\gamma m\beta c} x' - \frac{q\,B'}{2\gamma m\beta c} x = 0 \end{cases}$$

$B' \rightarrow \frac{dB}{dz} \rightarrow$ Longitudinal field on axis

- Larmor frequency and Larmor frame [1]
  - ✓ $\omega_L = \frac{qB}{2\gamma m}$
  - ✓ A frame rotating with Larmor frequency.

- Equation of motion in Larmor frame.

$$\begin{cases} x_L'' + k_0^2(z) x_L = 0 \\ y_L'' + k_0^2(z) y_L = 0 \end{cases} , \quad k_0(z) = \frac{qB(z)}{2\gamma m\beta c} \quad \Longrightarrow \quad \text{focusing in both direction}$$

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[1] M. Reiser, "Theory and Design of Charge Particle Beams", 2nd edition, Wiley-VCH, 2008.(chapter 3).

RF defocusing in a TW structure

- Field expansions

$$\begin{cases} E_z(r,z,t) = E_0^{TW} \sum_{n=-\infty}^{\infty} C_n^{(TW)} J_0(K_n r) e^{i(\omega t - k_n z)} \\ E_r(r,z,t) = iE_0^{TW} \sum_{n=-\infty}^{\infty} C_n^{(TW)} \frac{k_n}{K_n} J_1(K_n r) e^{i(\omega t - k_n z)} \\ B_\varphi(r,z,t) = iE_0^{TW} \sum_{n=-\infty}^{\infty} C_n^{(TW)} \frac{k}{K_n c} J_1(K_n r) e^{i(\omega t - k_n z)} \end{cases}$$

- Principle wave dominated structure

$$F_r = -e(E_r - \beta c B_\theta) = e\frac{\omega}{\beta_w c}(1-\beta\beta_w)\frac{J_1(K_0 r)}{K_0} E_0 \sin\theta \quad , \quad K_0 = \frac{i\omega}{c\beta_{ph}\gamma_{ph}}$$

- Paraxial approximation ($K_0 r \ll 1 \Longrightarrow J_1(K_0 r) \cong \frac{K_0 r}{2}$)

$$\Longrightarrow \qquad F_r = \frac{e\omega E_0}{2c}\frac{(1-\beta\beta_{ph})}{\beta_{ph}}\sin\theta\, r$$

⟶ Linrear

⟶ Phase dependent focusing

⟶ Zero for on crest acceleration

⟶ Zero for relativistic beams

Envelope equation

- From particle trajectory to beam envelope

(1)

$$x'' + kx = 0$$

(2) (3)

$$a^2 = \langle x^2 \rangle \quad \rightarrow \quad aa' = \langle xx' \rangle \quad \rightarrow \quad a'^2 + aa'' = \langle x'^2 \rangle + \langle xx'' \rangle$$

(1) (2) (3)

$$\Longrightarrow a'' + ka - \frac{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}{a^3} = 0$$

$\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \rightarrow \varepsilon_g^2$

- Including the space charge [2] (valid for beams of elliptical symmetry)

$$a'' + ka - \frac{\varepsilon_g^2}{a^3} - \frac{K}{4a} = 0 \quad , \quad K = \frac{qI}{4\pi\epsilon_0 mc^3} \frac{2}{\beta^3 \gamma^3}$$

  - ✓ Emittance dominated beam: $\frac{\varepsilon_g^2}{a^3} \gg \frac{K}{4a}$
  - ✓ Space charge dominated beam: $\frac{\varepsilon_g^2}{a^3} \ll \frac{K}{4a}$

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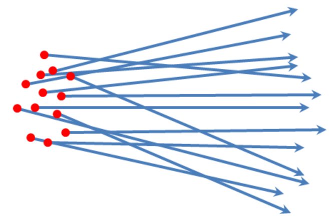
[1] Sacherer, F. J., “RMS Envelope Equations with Space Charge”, IEEE Transactions on Nuclear Science, Volume 18 Issue 3, 1971.

4. Emittance growth

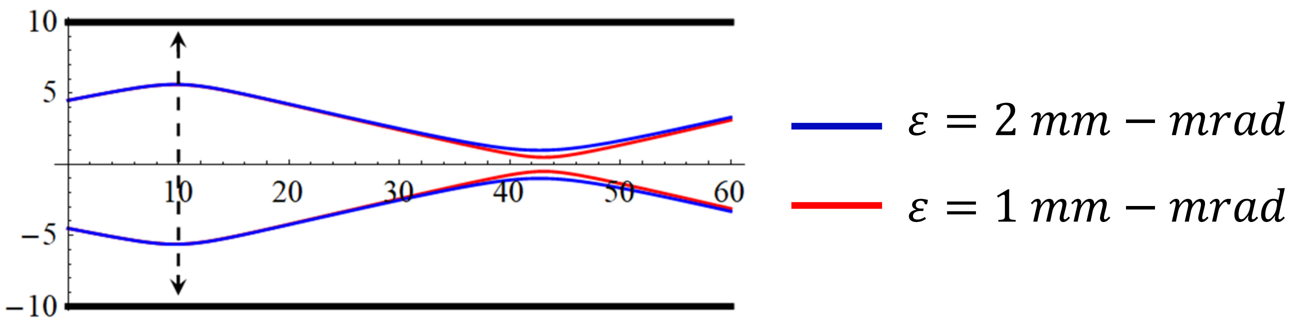
Emittance and the concept of beam quality

- A quantitative measure of the beam quality

$$a'' + ka - \frac{\varepsilon_g^2}{a^3} - \frac{K}{4a} = 0$$



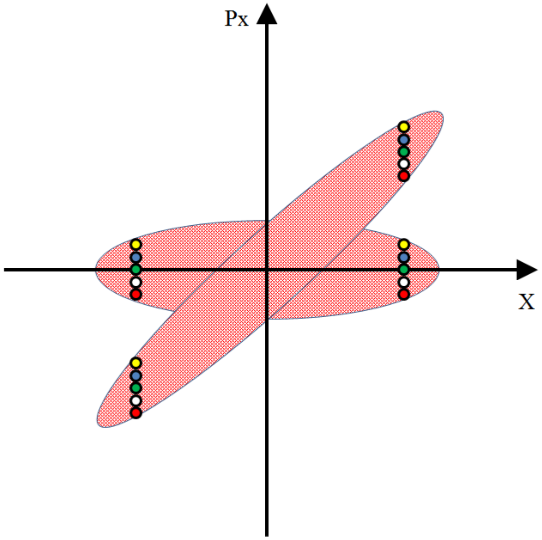
- Beam size at waist:
  - ✓ A beam of larger emittance has a larger size at waist.



- Focusing strength:
  - ✓ The focusing strength is approximately proportional to the beam emittance.

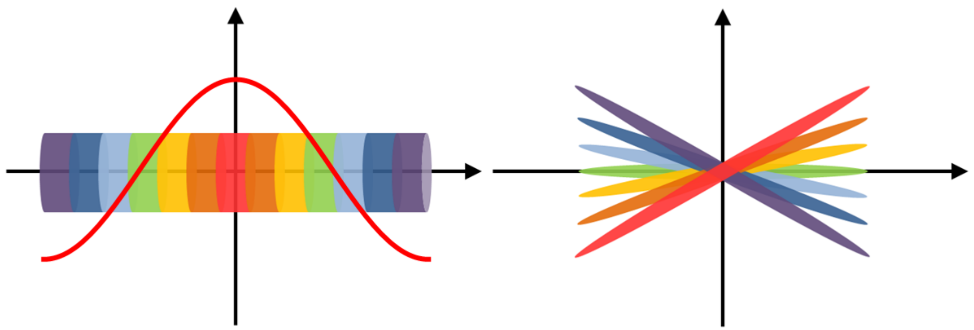
Emittance growth sources

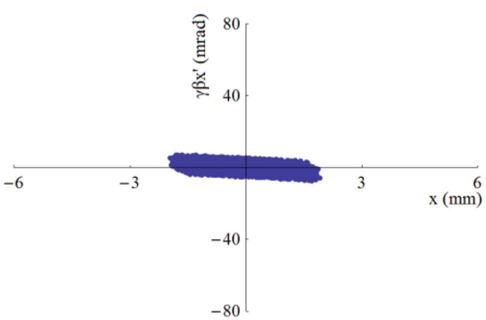
- Transverse RF forces
  - ✓ Time dependent forces violating Liouville's theorem allow for the emittance growth.
  - ✓ Focusing in a magnetostatic lens.
    - Particles at the same position receive the same focusing impulse resulting in a conserved phase space area.



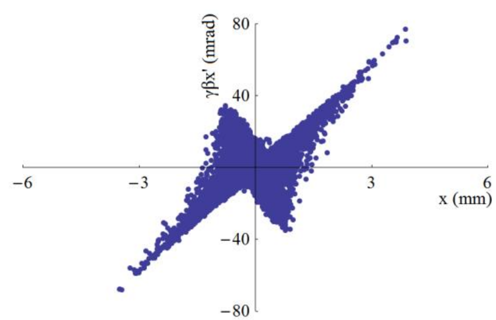
Emittance growth sources

- Transverse RF forces
  - ✓ RF defocusing
    - $F_r = \frac{e\omega E_0}{2c}\frac{(1-\beta\beta_{ph})}{\beta_{ph}}\sin\theta\, r$ $\longrightarrow$ a phase dependent focusing.



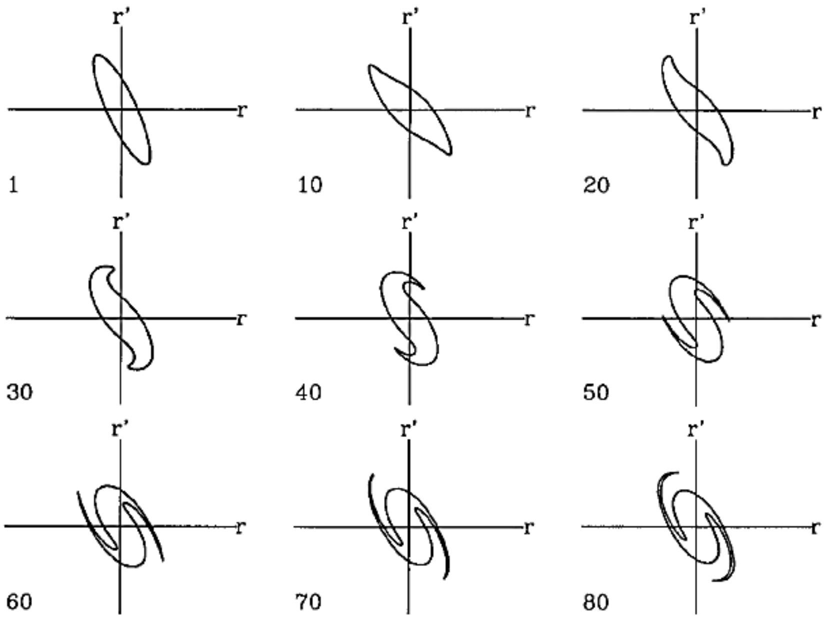


After the first cell of the buncher ⟹



Emittance growth sources

- Nonlinear forces
    - ✓ Filamentation: the area of the phase space is conserved but the shape is not



    - ✓ The phase space area is not the best quantity describing the beam quality.

$\Longrightarrow$ rms emittance

Emittance growth sources

- linear forces

$$\varepsilon_g^2 = \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2$$

$$\frac{d\varepsilon_g^2}{dz} = 2\langle xx'\rangle\langle x'^2\rangle + 2\langle x^2\rangle\langle x'x''\rangle - 2\langle xx'\rangle(\langle x'^2\rangle + \langle xx''\rangle)$$
$$= 2\langle x^2\rangle\langle x'x''\rangle - 2\langle xx'\rangle\langle xx''\rangle$$

$$x'' + kx = 0 \quad \Longrightarrow \quad \begin{cases}\langle x'x''\rangle = -k\langle xx'\rangle \\ \langle xx''\rangle = -k\langle x^2\rangle\end{cases}$$

$$\Longrightarrow \quad \frac{d\varepsilon_g^2}{dz} = 0$$

5. Optical functions

Optical functions

- An equivalent approach to the envelope equation describing the envelope evolution when the emittance is conserved. A matrix method instant of solving a nonlinear deferential equation.

- Definitions

$$\beta = \frac{\langle x^2 \rangle}{\varepsilon_g} \qquad \text{or} \qquad a = \sqrt{\beta \varepsilon_g}$$

$$\alpha = -\frac{1}{2}\beta' = -\frac{\langle x x\prime \rangle}{\varepsilon_g}$$

$$\gamma = \frac{1-\alpha^2}{\beta} = \langle x'^2 \rangle$$

- Transfer matrix

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

$$X = MX_0$$

Optical functions

➢ The evolution of the optical functions

$$X = MX_0 \quad \rightarrow \quad X^T = X_0^T M^T$$

$$XX^T = MX_0X_0^T M^T \xrightarrow{averaging} \langle XX^T \rangle = M\left\langle X_0X_0^T \right\rangle M^T$$

$B = \langle XX^T \rangle \quad \longrightarrow \quad$ definition of beta matrix

$$B = \left\langle \begin{bmatrix} x \\ x' \end{bmatrix} \begin{bmatrix} x & x' \end{bmatrix} \right\rangle = \begin{bmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{bmatrix}$$

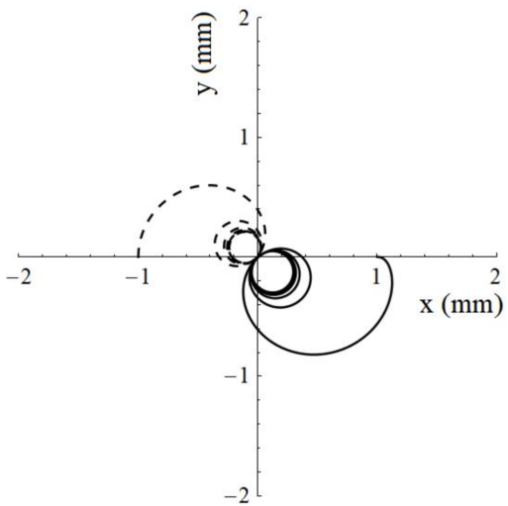
$$= \begin{bmatrix} \beta\varepsilon_g & -\alpha\varepsilon_g \\ -\alpha\varepsilon_g & \gamma\varepsilon_g \end{bmatrix}$$

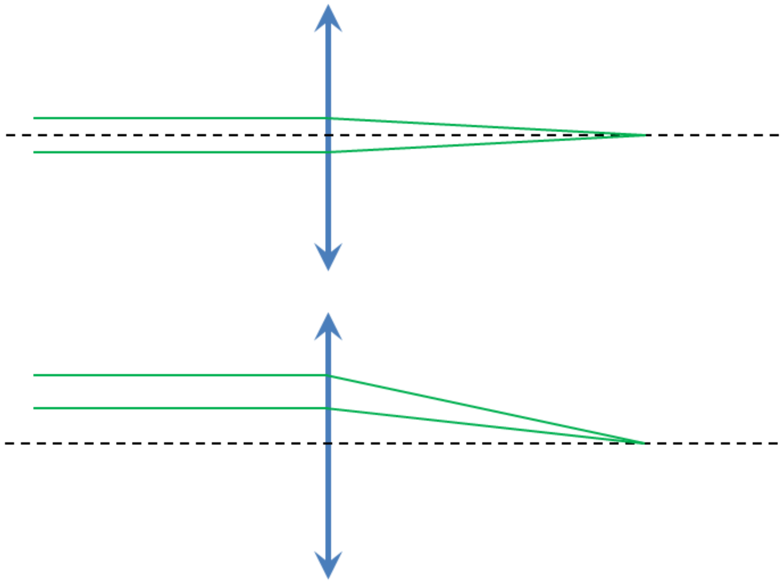
If $\varepsilon_g = \varepsilon_{0g} \longrightarrow$ $\boxed{\begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix} = M \begin{bmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{bmatrix} M^T}$

6. Alignment

Alignment

- Misalignment of a focusing element produce a kick. The beam is drawn an average toward the axis.

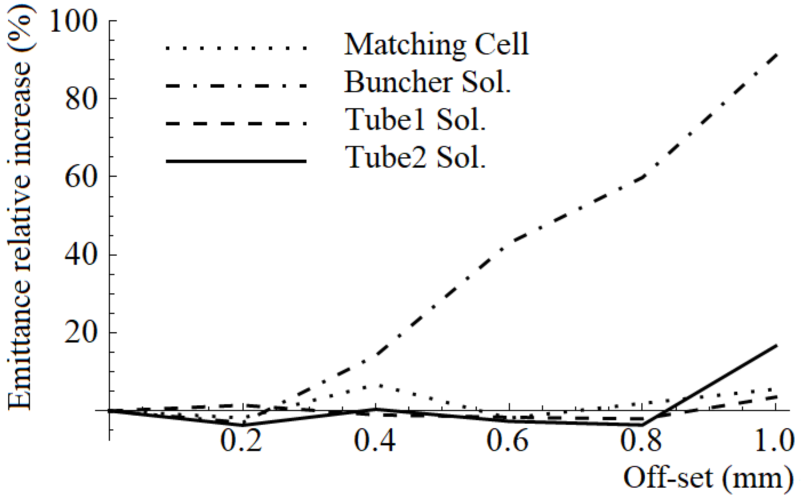




- Stanford Linear Collider (3.2 km length Linac) was claimed to be the world's most straight object.

Misalignment

- The kick results in beam loss
  - ✓ In our linac: $\Delta x = 2m \tan 1° \cong 3.5\ cm$!
- In a RF cavity
  - ✓ With a beam off-set particles travel on a further distance with respect to the cavity axis (on average) experiencing a larger RF defocusing and hence a larger emittance growth.



Thanks for your attention!