Introduction to Relativistic Heavy-Ion Collisions
Lecture 10: Freeze-out modelling

Radoslaw Ryblewski
The H. Niewodniczański Institute of Nuclear Physics
Polish Academy of Sciences
Kraków, Poland

NATIONAL SCIENCE CENTRE
POLAND
SONATA BIS 8 Grant No. 2018/30/E/ST2/00432

IPM
پژوهشگاه دانش‌های بنیادی

THE HENRYK NIEWODNICZAŃSKI
INSTITUTE OF NUCLEAR PHYSICS
POLISH ACADEMY OF SCIENCES

figure: https://www.bnl.gov/newsroom

Fluid dynamics from kinetic theory

The evolution of $f(x,p)$ follows from the standard relativistic Boltzmann equation

$$p^\alpha \partial_\alpha f = -C[f]$$

Often the collisional kernel is treated in the relaxation-time approximation

$$C[f] = p_\mu u^\mu \frac{f - f_{\mathrm{eq}}}{\tau_{\mathrm{eq}}}$$

Equations of motion for the soft modes of the system, identified with the fluid dynamical sector of the theory, may be derived by taking the lowest-*n* momentum moments

$$\hat{\mathscr{I}}^{\mu_1 \cdots \mu_n} \equiv \int dP \, p^{\mu_1} p^{\mu_2} \cdots p^{\mu_n}, \qquad \int dP \equiv \int \frac{d^3 p}{E_p},$$

of the Boltzmann equation which gives

$$\partial_\alpha \mathscr{I}^{\alpha \mu_1 \cdots \mu_n} = -\mathscr{C}^{\mu_1 \cdots \mu_n}[f]$$

$$\mathscr{I}^{\alpha \mu_1 \cdots \mu_n} \equiv \hat{\mathscr{I}}^{\alpha \mu_1 \cdots \mu_n} f,$$

$$\mathscr{C}^{\mu_1 \cdots \mu_n}[f] \equiv \hat{\mathscr{I}}^{\mu_1 \cdots \mu_n} C[f]$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} \partial_\mu \mathscr{I}^\mu &= u_\mu \frac{\mathscr{I}^\mu_{\mathrm{eq}} - \mathscr{I}^\mu}{\tau_{\mathrm{eq}}}, & \qquad N^\mu &\equiv \mathscr{I}^\mu \\ \partial_\mu \mathscr{I}^{\mu\nu} &= u_\mu \frac{\mathscr{I}^{\mu\nu}_{\mathrm{eq}} - \mathscr{I}^{\mu\nu}}{\tau_{\mathrm{eq}}}. & \qquad T^{\mu\nu} &\equiv \mathscr{I}^{\mu\nu} \end{aligned}}$$

Fluid dynamics from kinetic theory

$$\hat{\mathscr{I}}^{\mu_1\cdots\mu_n} \equiv \int dP\, p^{\mu_1} p^{\mu_2} \cdots p^{\mu_n}, \qquad \int dP \equiv \int \frac{d^3p}{E_p},$$

$$\mathscr{I}^{\alpha\mu_1\cdots\mu_n} \equiv \hat{\mathscr{I}}^{\alpha\mu_1\cdots\mu_n} f,$$

$$N^\mu \equiv \mathscr{I}^\mu$$

$$T^{\mu\nu} \equiv \mathscr{I}^{\mu\nu}$$

$$f_{\rm eq} = f(p_\mu u^\mu, T, \mu_{\rm i})$$

$$T^{\mu\nu} = \mathscr{E} u^\mu u^\nu - \mathscr{P}\Delta^{\mu\nu}$$

$$N_i^\mu = \mathscr{N}_i u^\mu$$

$\mathscr{E} = \mathscr{E}(T,\mu_{\rm i})$, $\mathscr{P} = \mathscr{P}(T,\mu_{\rm i})$, and $\mathscr{N} = \mathscr{N}(T,\mu_{\rm i})$

$$\partial_\mu \mathscr{I}^\mu = u_\mu \frac{\mathscr{I}^\mu_{\rm eq} - \mathscr{I}^\mu}{\tau_{\rm eq}},$$

$$\partial_\mu \mathscr{I}^{\mu\nu} = u_\mu \frac{\mathscr{I}^{\mu\nu}_{\rm eq} - \mathscr{I}^{\mu\nu}}{\tau_{\rm eq}}.$$

$$u_\mu(\mathscr{I}^\mu_{\rm eq} - \mathscr{I}^\mu) = 0$$

$$u_\mu(\mathscr{I}^{\mu\nu}_{\rm eq} - \mathscr{I}^{\mu\nu}) = 0$$

$$\partial_\mu N_i^\mu(x) = 0 \qquad (i = 1 \ldots N)$$

$$\partial_\mu T^{\mu\nu}(x) = 0$$

$Q_i = \{B, I_3, S, C\}$, where B, I_3, S and C

Fluid dynamics from kinetic theory

$$T^{\mu\nu} = \mathscr{E} u^\mu u^\nu - \mathscr{P} \Delta^{\mu\nu}$$

$$N_i^\mu = \mathscr{N}_i u^\mu$$

$\mathscr{E} = \mathscr{E}(T, \mu_{\rm i})$, $\mathscr{P} = \mathscr{P}(T, \mu_{\rm i})$, and $\mathscr{N} = \mathscr{N}(T, \mu_{\rm i})$

$$\partial_\mu N_i^\mu(x) = 0 \qquad (i = 1 \ldots N)$$

$$\partial_\mu T^{\mu\nu}(x) = 0$$

$Q_i = \{B, I_3, S, C\}$, where B, I_3, S and C

$$\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} = (\mathscr{E} + \mathscr{P}) D u^\alpha - \nabla^\alpha \mathscr{P} = 0,$$
$$u_\nu \partial_\mu T^{\mu\nu} = D\mathscr{E} + (\mathscr{E} + \mathscr{P})\theta = 0,$$
$$\partial_\mu N_i^\mu = D\mathscr{N}_i + \mathscr{N}_i \theta = 0$$
$$\mathscr{P} = \mathscr{P}(\mathscr{E}, \mathscr{N}_i)$$

$$u^\mu(x) \equiv \gamma(1, \mathbf{v}_T, v_z)$$

$$u^\mu = (u_0 \cosh y_u, \mathbf{u}_T, u_0 \sinh y_u)$$

$$u_0 = \sqrt{1 + u_T^2}$$

Equation of state

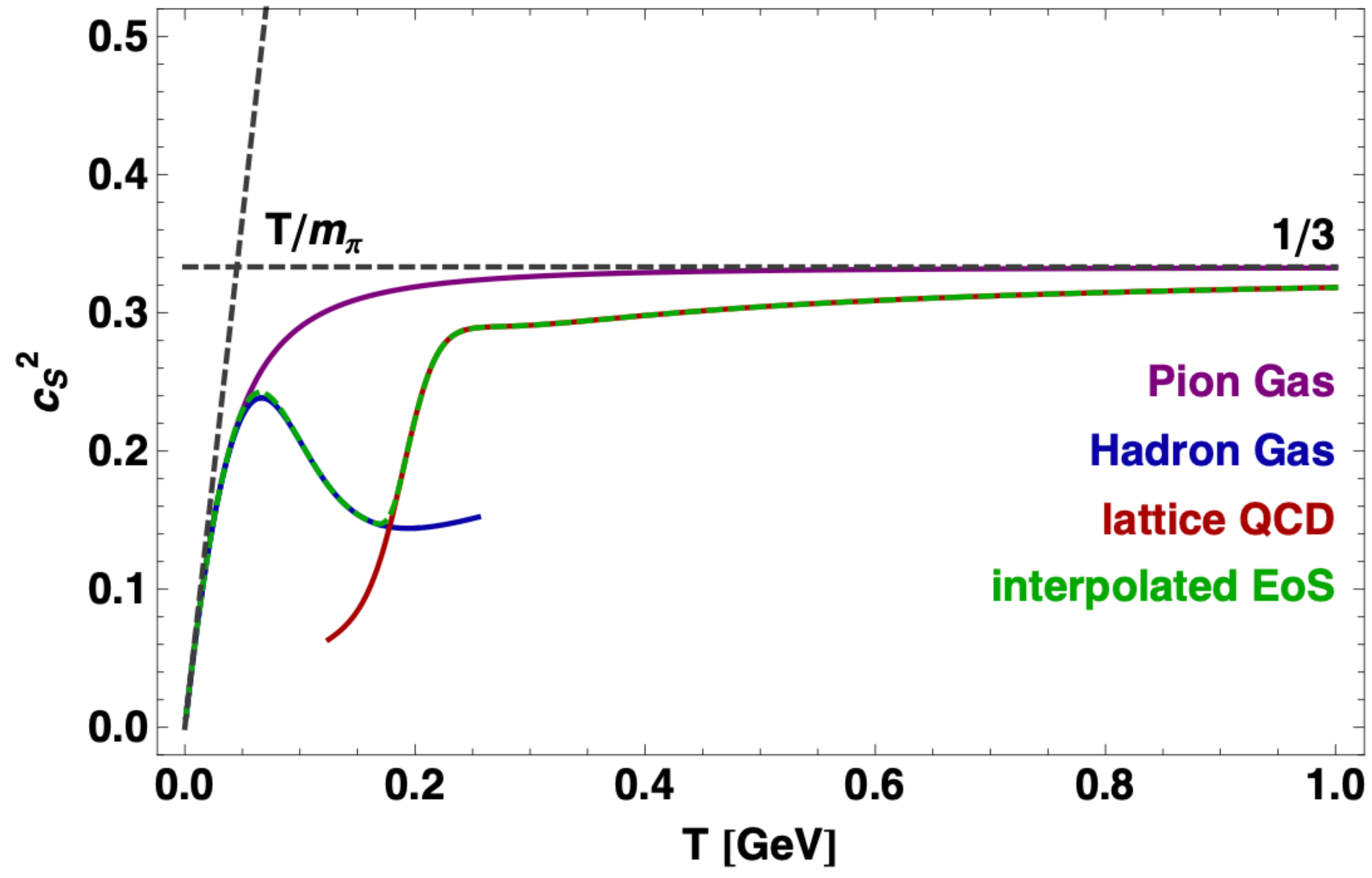
$$\mathscr{P} = \mathscr{P}(\mathscr{E}, \mathscr{N}_i)$$

Or

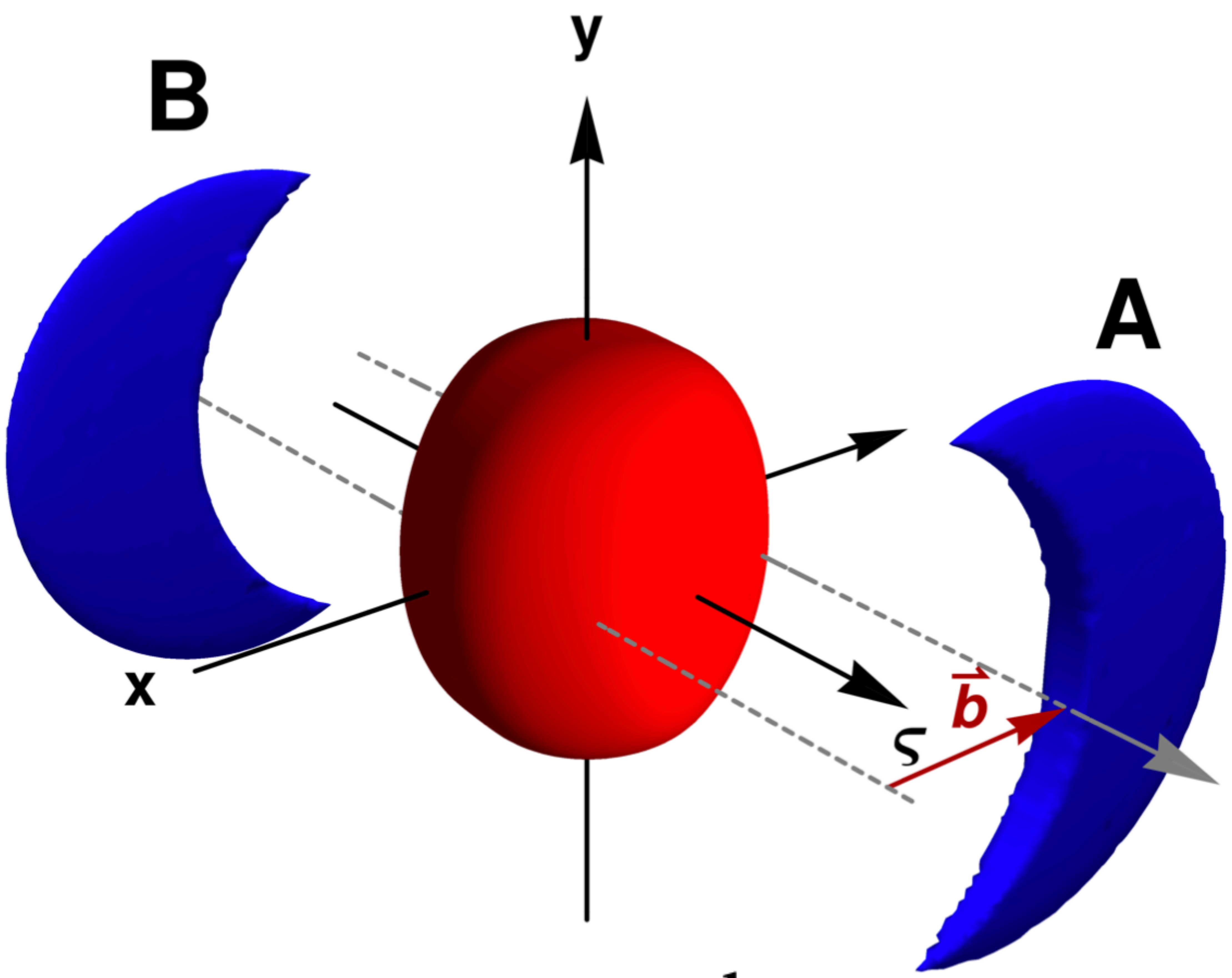
$$\mathscr{E} = \mathscr{E}(T, \mu_\mathrm{i}),\ \mathscr{P} = \mathscr{P}(T, \mu_\mathrm{i}),\ \text{and}\ \mathscr{N} = \mathscr{N}(T, \mu_\mathrm{i})$$

For the charge-free matter in equilibrium energy density may be directly related to the temperature of the system

$$\mathscr{E} = \mathscr{E}(T), \mathscr{P} = \mathscr{P}(T)$$
$$c_s^2(T) = d\mathscr{P}/d\mathscr{E}$$



Collision geometry



$$t = \tau \cosh \varsigma,$$

$$z = \tau \sinh \varsigma,$$

Initial conditions for hydrodynamics

EOMs have to be supplemented with proper initial conditions, specified on the hypersurface of constant longitudinal proper time $\tau = \tau_{\mathrm{i}}$.

$$\mathscr{E}(\tau_{\mathrm{i}}, \mathbf{x}_T, \varsigma),\, u_x(\tau_{\mathrm{i}}, \mathbf{x}_T, \varsigma),\, u_y(\tau_{\mathrm{i}}, \mathbf{x}_T, \varsigma),\, y_u(\tau_{\mathrm{i}}, \mathbf{x}_T, \varsigma), \text{ and } \mathscr{N}_i(\tau_{\mathrm{i}}, \mathbf{x}_T, \varsigma)$$

Tilted source model

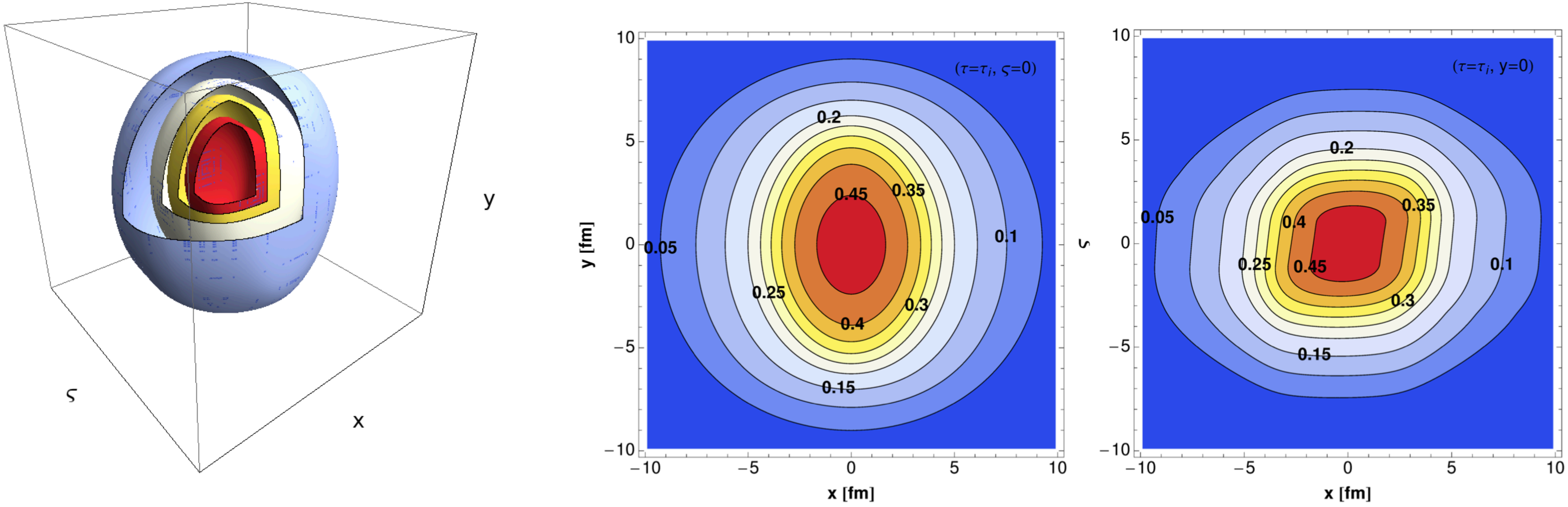
$$\mathscr{E}(\tau_{\mathrm{i}}, \mathbf{x}_T, \varsigma, b) = \mathscr{E}_{\mathrm{i}} \frac{n(\mathbf{x}_T, \varsigma, b)}{n(\mathbf{0}, 0, 0)},$$

$$n(\mathbf{x}_T, \varsigma, b) = G(\varsigma) \left\{ (1-\kappa) \Big[W_A(\mathbf{x}_T, b) F(\varsigma) + W_B(\mathbf{x}_T, b) F(-\varsigma) \Big] + \kappa B(\mathbf{x}_T, b) \right\}.$$

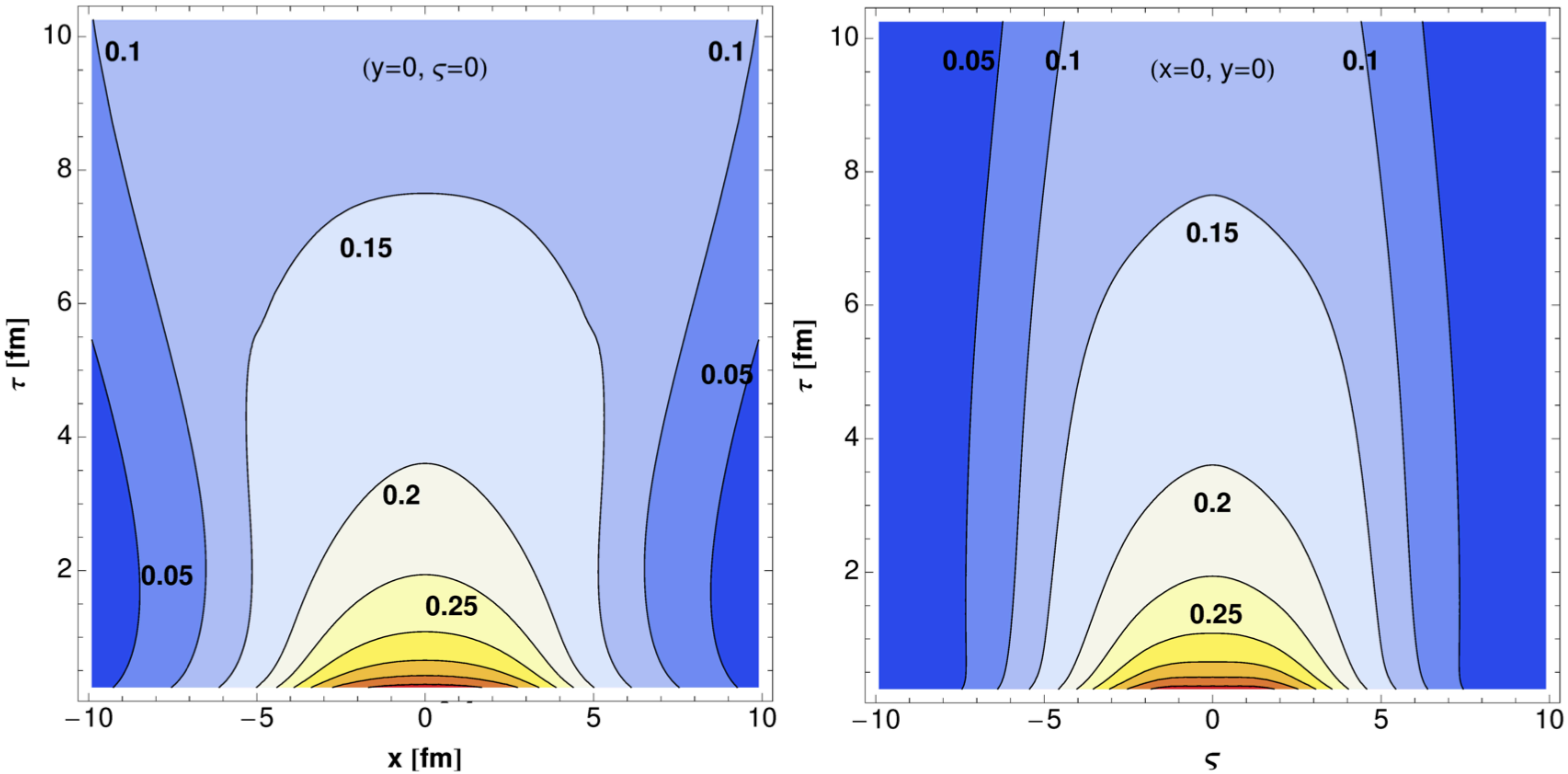
$$G(\varsigma) \equiv \exp\left[-\frac{(\varsigma - \Delta\varsigma)^2}{2\sigma_\varsigma^2} \Theta(|\varsigma| - \Delta\varsigma) \right]$$

$$F(\varsigma) = \begin{cases} 0 & \\ (\varsigma + y_{\mathrm{N}})/(2y_{\mathrm{N}}) & \text{if} \\ 1 & \end{cases} \quad \begin{array}{c} \varsigma < -y_{\mathrm{N}}, \\ -y_{\mathrm{N}} \le \varsigma \le y_{\mathrm{N}}, \\ \varsigma > y_{\mathrm{N}}, \end{array}$$

Initial conditions for hydrodynamics

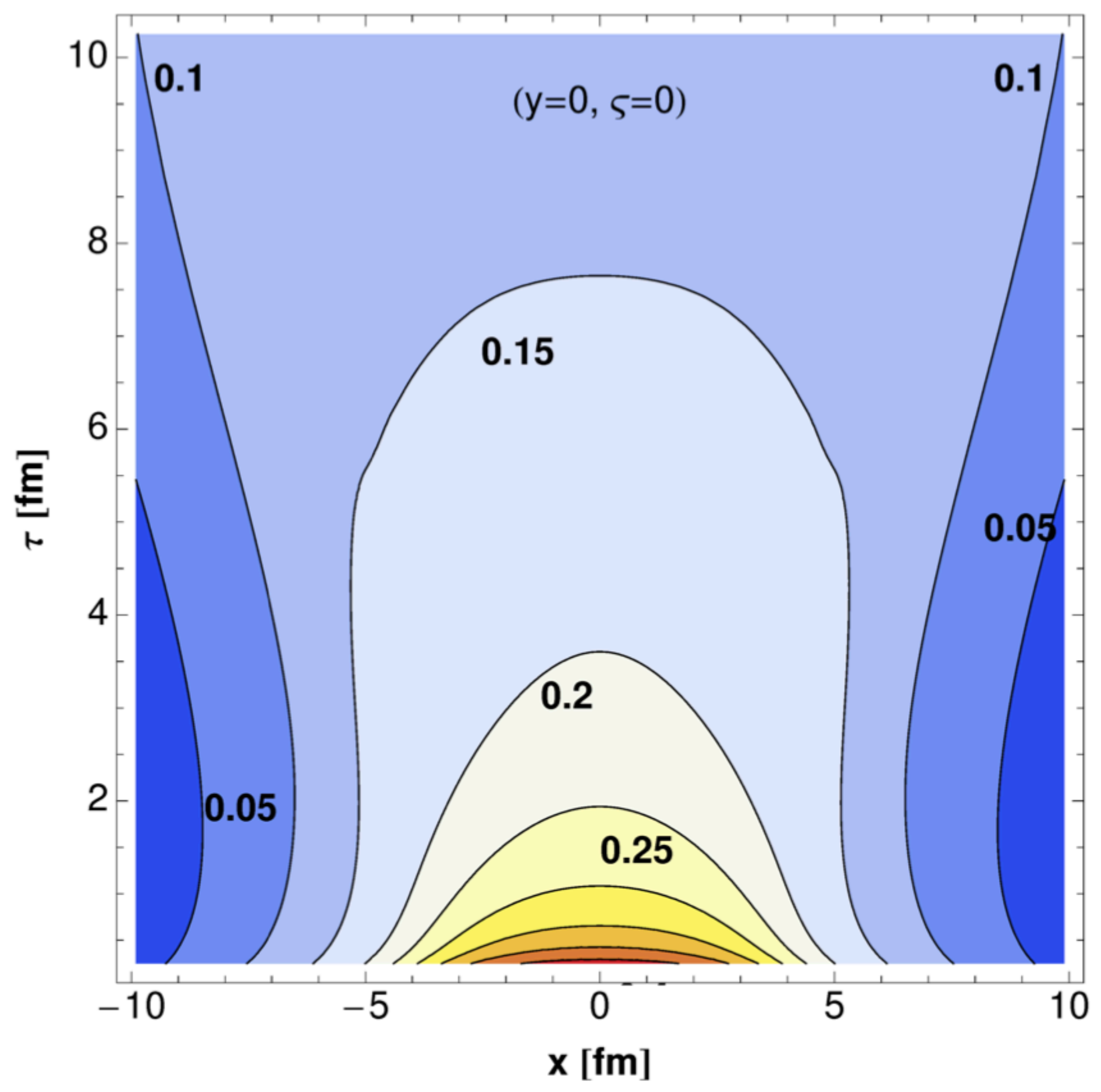


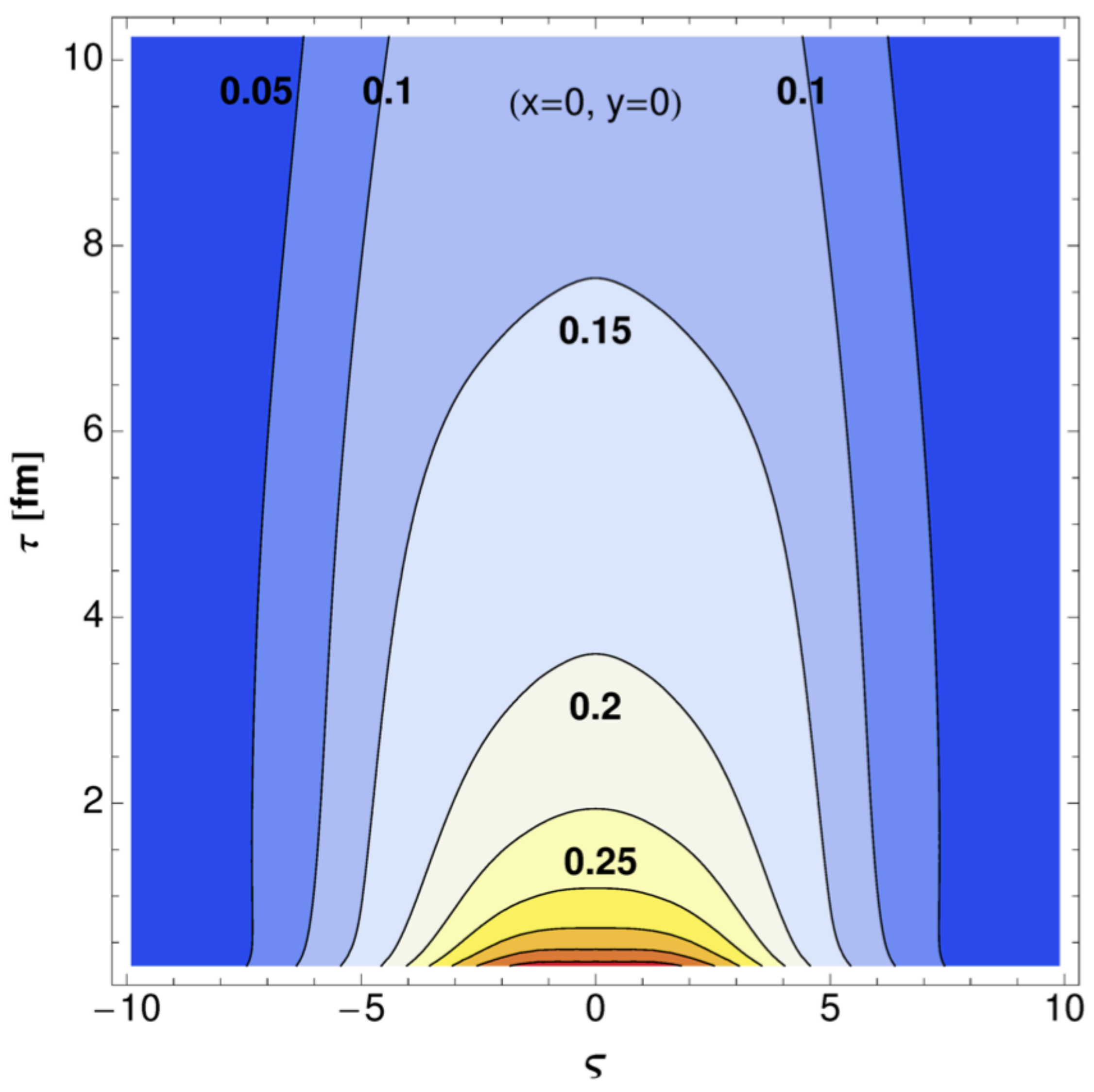
Evolution and decoupling

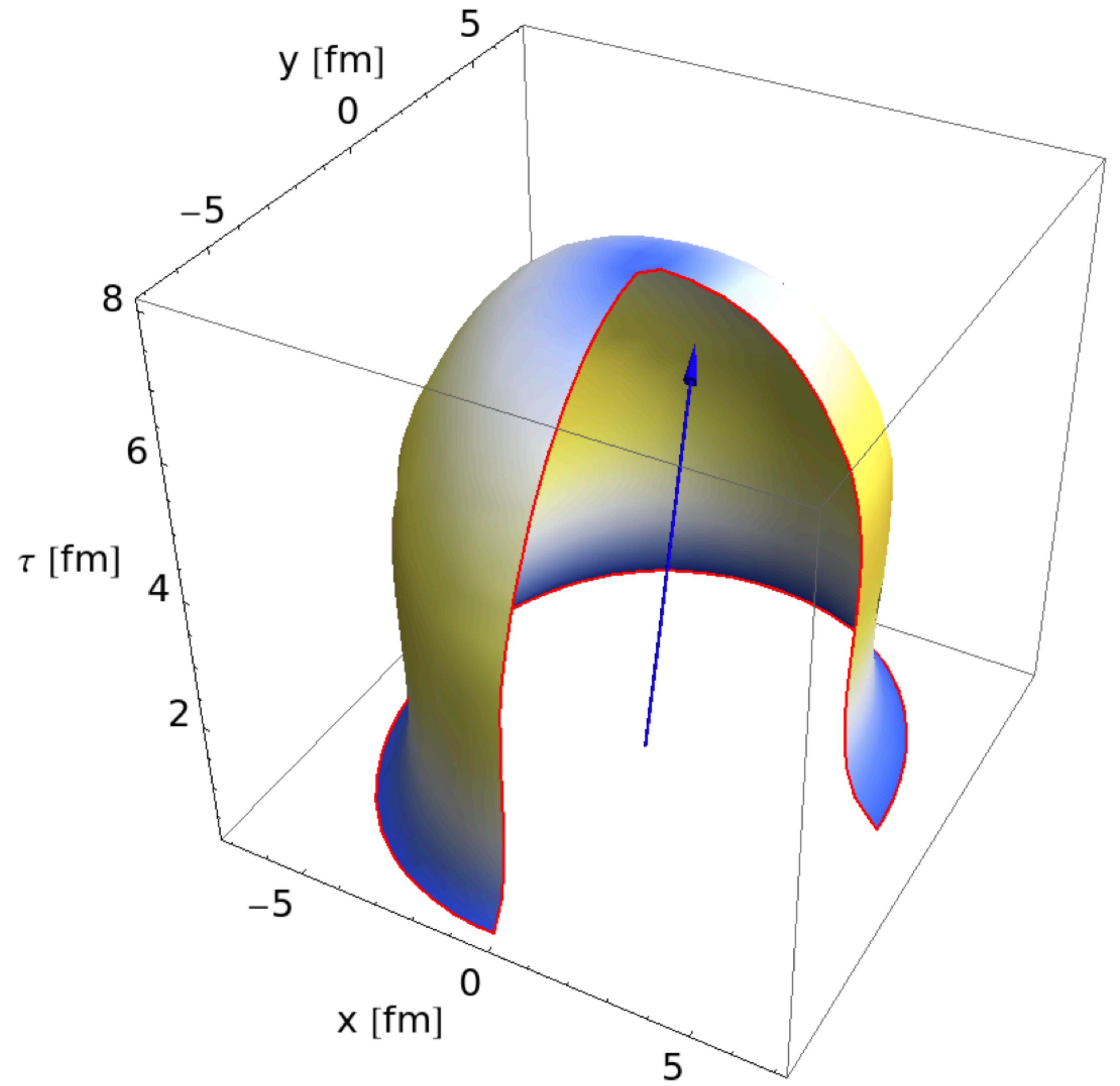


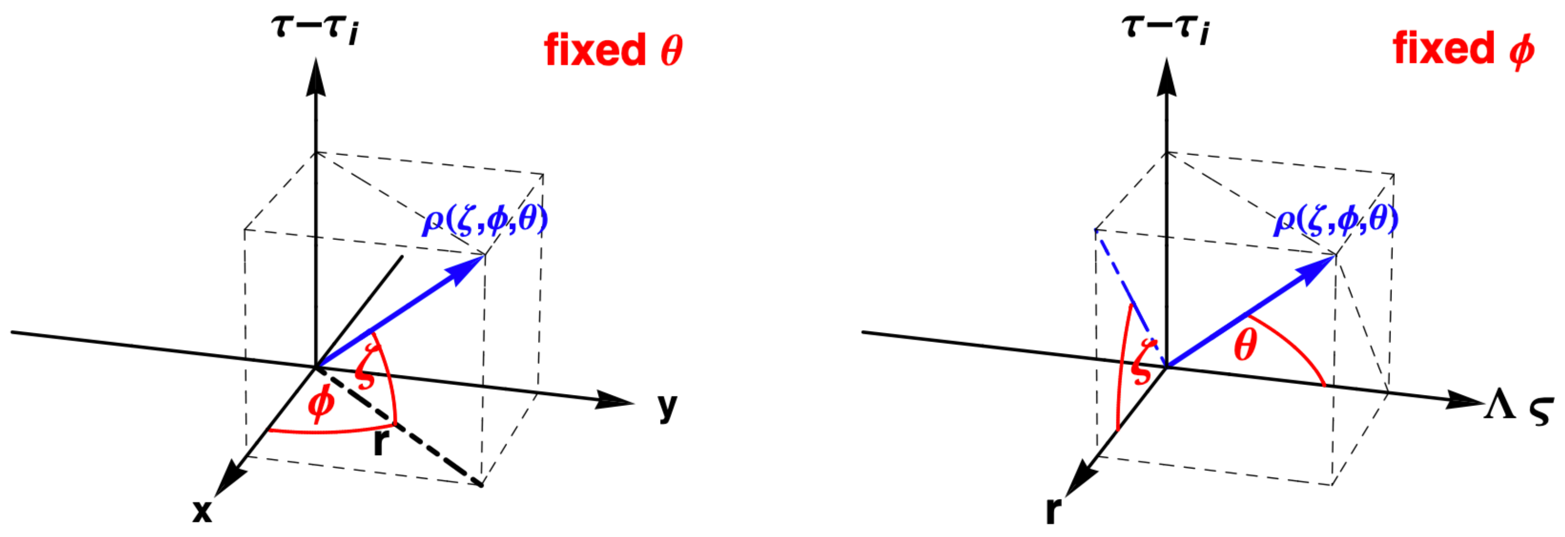
Eventually, the particle scatterings become too rare to prevent the particles from leaving the fluid. As a result the local thermal equilibrium cannot be maintained anymore and the fluid description breaks down. This complicated gradual process of particle decoupling from the fluid is often called the freeze-out

Freeze-out hypersurface extraction









$$x^0 = t(\rho,\theta,\zeta,\phi) = \tau\cosh\varsigma,$$
$$x^1 = x(\rho,\theta,\zeta,\phi) = r\sin\theta\cos\phi,$$
$$x^2 = y(\rho,\theta,\zeta,\phi) = r\sin\theta\sin\phi,$$
$$x^3 = z(\rho,\theta,\zeta,\phi) = \tau\sinh\varsigma,$$

$$\Lambda\,\varsigma = \rho\cos\theta,$$
$$r = \rho\cos\zeta,$$
$$\tau - \tau_i = \rho\sin\theta\sin\zeta$$

$$0 \le \zeta \le \pi/2, \qquad 0 \le \phi < 2\pi, \qquad \text{and} \qquad 0 \le \theta \le \pi$$

$$\Sigma(\theta,\zeta,\phi) = \{x \in \mathscr{M} : T(\rho,\theta,\zeta,\phi) = T_{\rm freeze}\}$$

$$\rho = \rho(\theta,\zeta,\phi) \qquad u^\mu = u^\mu(\theta,\zeta,\phi)$$

$$d^3\Sigma_\mu = \varepsilon_{\mu\alpha\beta\gamma}\frac{\partial x^\alpha}{\partial\zeta}\frac{\partial x^\beta}{\partial\phi}\frac{\partial x^\gamma}{\partial\theta}d\zeta d\phi d\theta$$

Cooper-Frye formalism

In the transport theory framework the flux of particle species k is expressed with the formula

$$N_k^\mu(x) = \int \frac{d^3p}{E_p} p^\mu f_k(x,p)$$

The number of world-lines of particles of species k crossing the infinitesimal element dΣ of the surface Σ is then calculated from the expression

$$N_\Sigma^k \equiv d^3\Sigma_\mu N_k^\mu = \int \frac{d^3p}{E_p} d^3\Sigma_\mu p^\mu f_k(x,p)$$

The invariant momentum spectrum of particles produced at the surface element dΣ

$$E_p \frac{dN_\Sigma^k}{d^3p} = d^3\Sigma_\mu p^\mu f_k(x,p)$$

Therefore, the invariant momentum distribution of hadrons emitted on the entire freeze-out hypersurface Σ is given by the integral commonly known as the **Cooper-Frye formula**

$$\boxed{E_p \frac{dN^k}{d^3p} = \int_\Sigma d^3\Sigma_\mu p^\mu f_k(x,p)}$$

Cooper-Frye formalism

It is usually assumed that just before decoupling from the fluid, i.e., before the particles cross the switching surface Σ, they are in local thermal and chemical equilibrium such that the phase-space distributions of produced hadrons follow either Fermi–Dirac (a = −1) or Bose–Einstein (a = +1) distributions

$$\boxed{f_k(x,p) = f_k\left(p_\mu u^\mu(x), T(x), \tilde{\mu}_k\right) = \frac{g_k}{(2\pi)^3}\left\{\exp\left[\frac{p_\mu u^\mu(x) - \tilde{\mu}_k(x)}{T(x)}\right] - a\right\}^{-1}}$$

$$g_k = 2s_k + 1$$
$$\tilde{\mu}_k = \sum_i Q_i^k \mu_i$$

$$p^\mu = (m_T \cosh y_p, p_T \cos\phi_p, p_T \sin\phi_p, m_T \sinh y_p)$$

$$\begin{aligned} d^3\Sigma_\mu p^\mu = & \frac{\sin\theta\tau\rho^2}{\Lambda}\Bigg[\frac{\partial\rho}{\partial\zeta}\cos\zeta\left(p_T\sin\zeta\cos(\phi_p-\phi) - m_T\cos\zeta\cosh(y_p-\varsigma)\right) \\ & +\cos\zeta\sin\theta\left(\rho\sin\theta - \frac{\partial\rho}{\partial\theta}\cos\theta\right) \\ & \times\left(p_T\cos\zeta\cos(\phi_p-\phi) + m_T\sin\zeta\cosh(y_p-\varsigma)\right) \\ & +\cos\zeta\sin\theta\left(\rho\cos\theta + \frac{\partial\rho}{\partial\theta}\sin\theta\right)\frac{\Lambda}{\tau}m_T\sinh(y_p-\varsigma) \\ & -\frac{\partial\rho}{\partial\phi}\,p_T\,\sin(\phi_p-\phi)\Bigg]d\zeta d\phi d\theta \equiv \boxed{h_k(\zeta,\phi,\theta,p_T,\phi_p,y_p)}d\zeta d\phi d\theta, \end{aligned}$$

$$\boxed{\begin{aligned} \frac{d^6N^k}{p_T dp_T d\phi_p dy_p d\zeta d\phi d\theta} &= \frac{g_k}{(2\pi)^3}h_k(\zeta,\phi,\theta,p_T,\phi_p,y_p) f_k(\zeta,\phi,\theta,p_T,\phi_p,y_p) \\ &\equiv \mathscr{F}_k(\zeta,\phi,\theta,p_T,\phi_p,y_p) \end{aligned}}$$

$$p_\mu u^\mu = u_0 m_T \cosh(y_p - y_u) - p_T u_T \cos(\phi_p - \phi_u)$$

Hadron abundances

Total number of particles emitted on the entire hypersurface may be expressed as follows

$$N_\Sigma^k \equiv d^3\Sigma_\mu N_k^\mu = \int \frac{d^3p}{E_p} d^3\Sigma_\mu p^\mu f_k(x,p) \qquad N_i^\mu = \mathcal{N}_i u^\mu$$

$$N_k \equiv \int_\Sigma d^3\Sigma_\mu N_k^\mu = \int_\Sigma d^3\Sigma_\mu u^\mu(x) \mathcal{N}_k(T(x), \tilde{\mu}_k(x))$$

If T and μ are constant along Σ, the integral of the flow pattern on the freeze-out manifold factorizes with

$$V_{\text{eff}} \equiv \int_\Sigma d^3\Sigma_\mu u^\mu(x)$$

When one considers ratios of particle multiplicities of different species, say a and b, the factor Veff cancels out completely in the ratios

$$\frac{N_a}{N_b} = \frac{\mathcal{N}_a(T, \tilde{\mu}_k)}{\mathcal{N}_b(T, \tilde{\mu}_k)}$$

This gives rise to the wide variety of analyses under the common name of thermal or statistical models. They focus mainly on the extraction of thermodynamic properties of the matter at the chemical freeze-out based on thermal analysis of the multiplicities of the experimentally measured particles and ratios thereof.

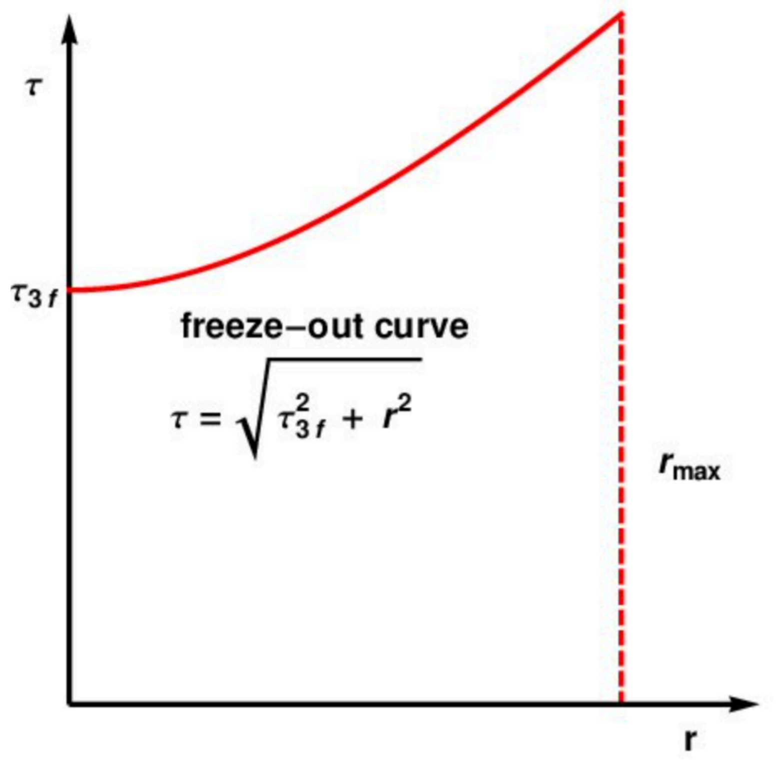
Hydro-inspired freeze-out models

Within these models the conditions at the freeze-out hypersurface as well as the shape of the freeze-out surface Σ(x), are simply assumed, or inspired, by the full numerical fluid dynamical simulations.
Among these models the most successful ones are the Blast-wave and Cracow models

$$(T(x), u^{\mu}(x), \mu_i(x))$$

Within the Cracow model (as well as in Blast-wave model) it is assumed that the freeze-out takes place at the boost-invariant and cylindrically symmetric surface (in the transverse x − y plane).
The hypersurface Σ is defined by the requirement that the particles freeze-out at the surface of constant proper-time

$$\tilde{\tau}^2 = x^{\mu} x_{\mu} = t^2 - x^2 - y^2 - z^2 = \tau^2 - r^2 = \tilde{\tau}^2_{\text{freeze}} = \text{const.}$$

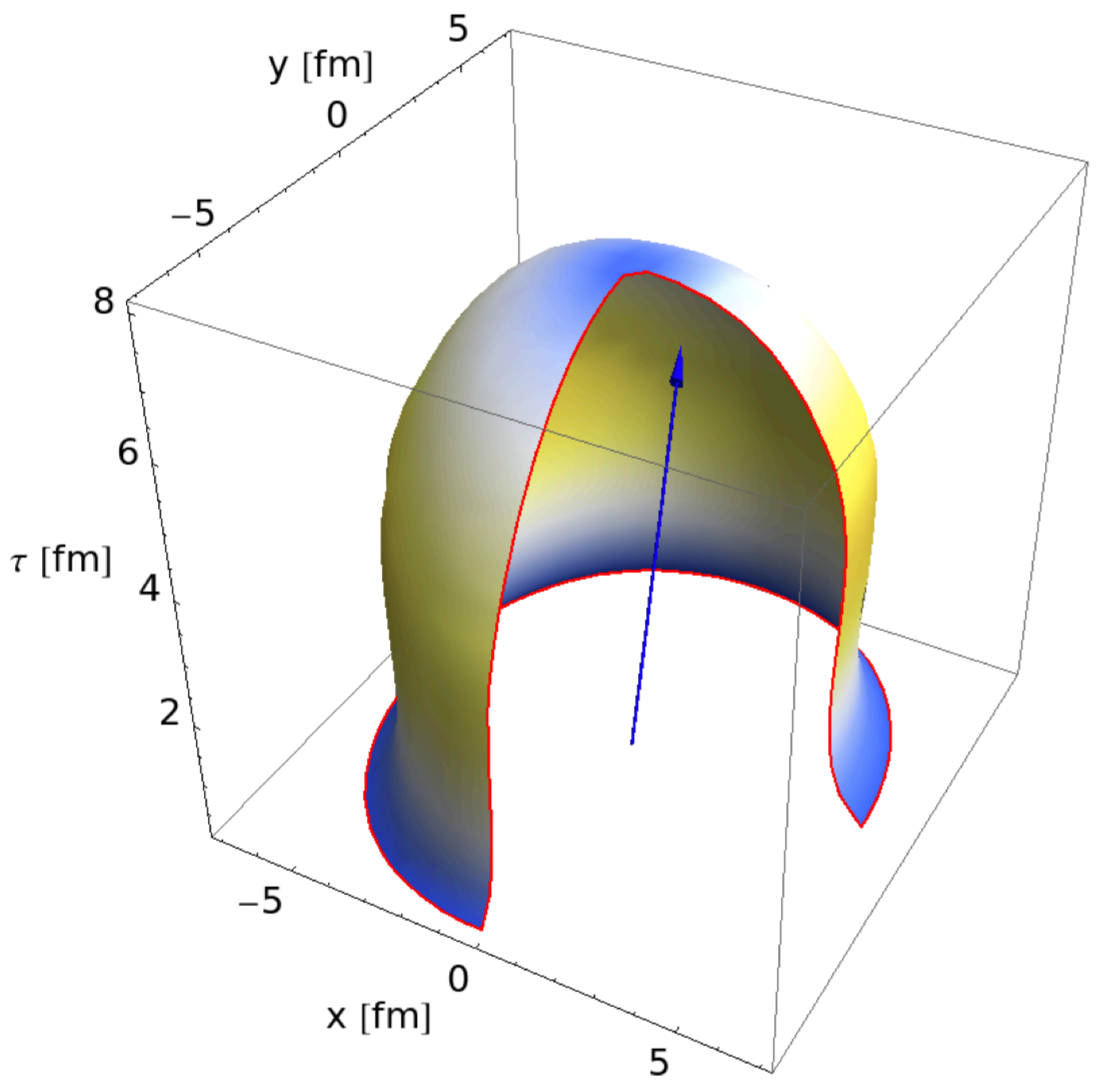


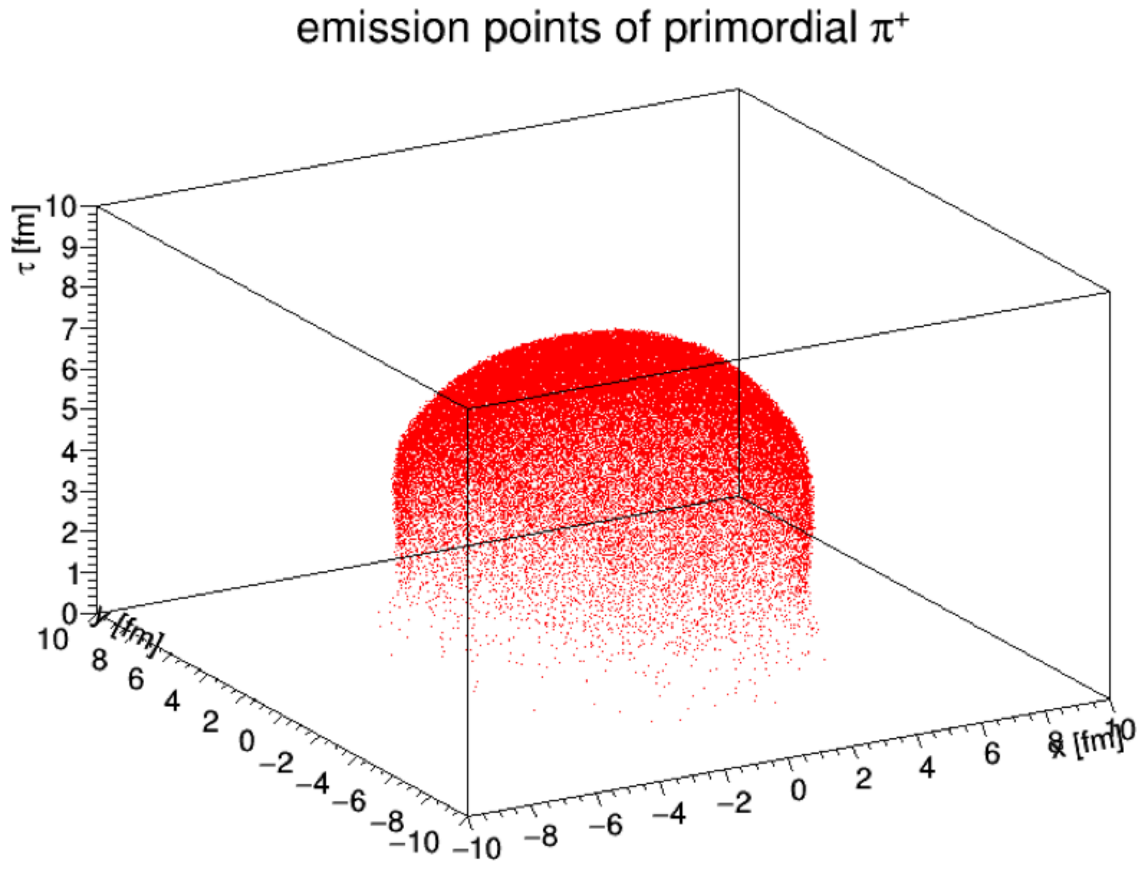
According to it the particles decouple starting from the center of the fire-cylinder towards its edge

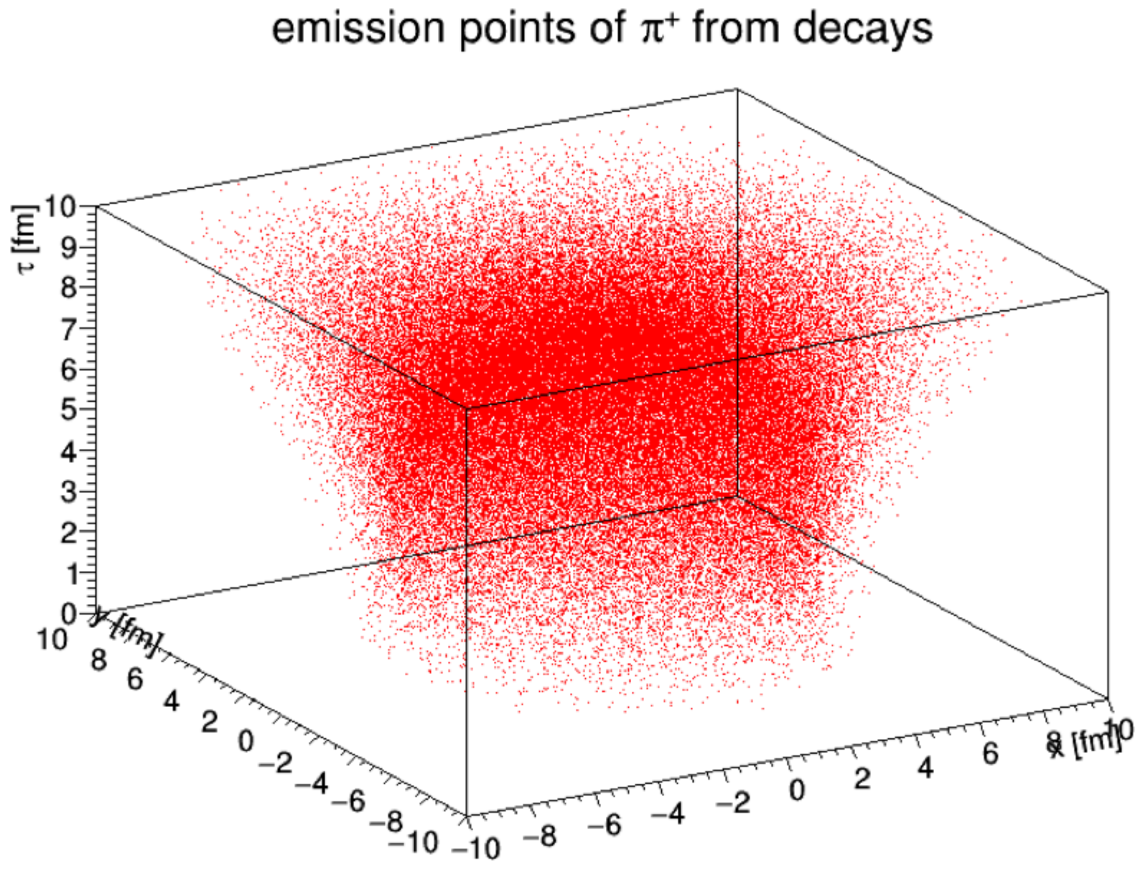
At the same time the fluid four-velocity is assumed to have the Hubble-like form

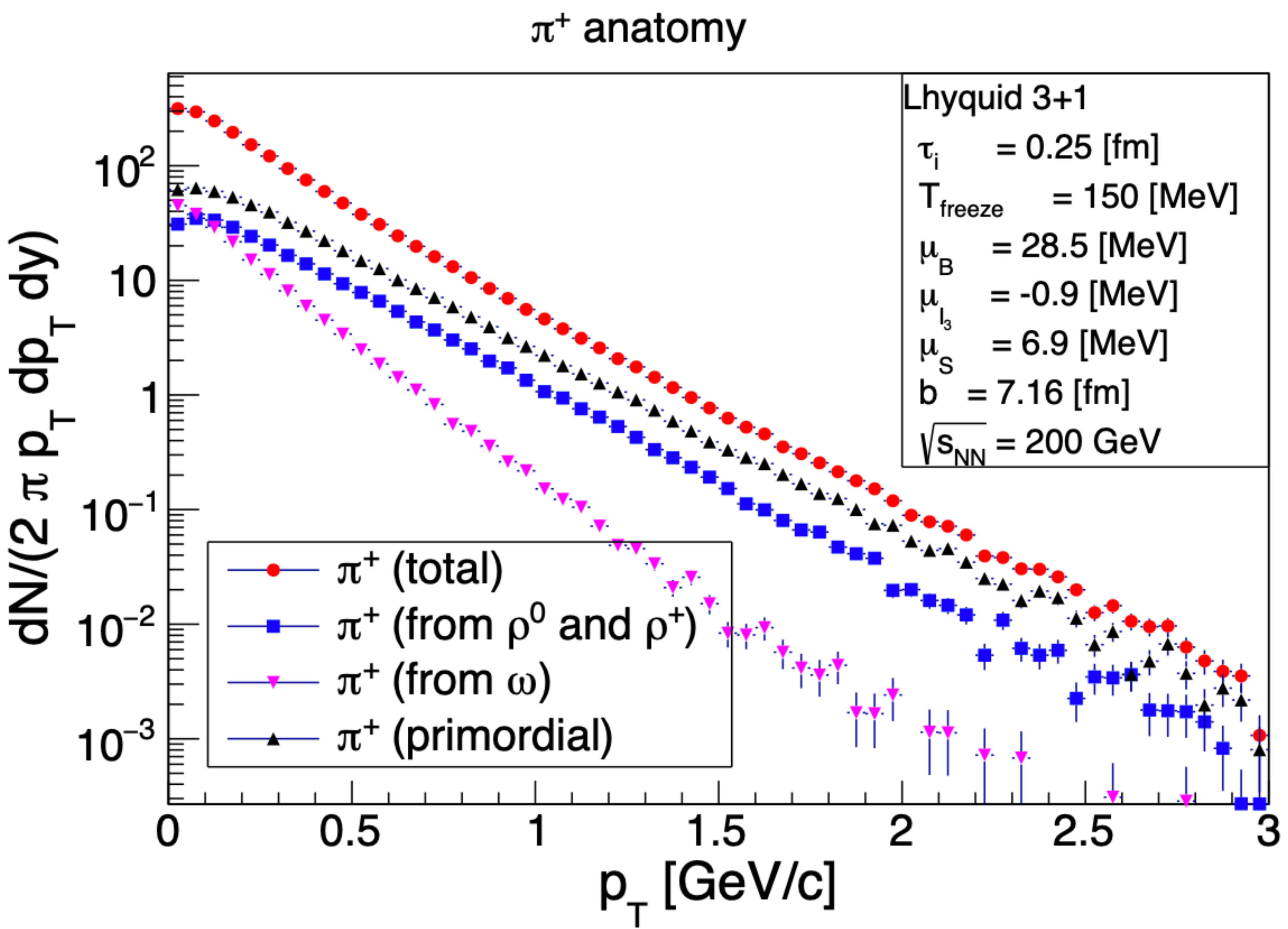
$$u^{\mu} = \gamma(1, \mathbf{v}_T, v_z) = \frac{x^{\mu}}{\tilde{\tau}_{\text{freeze}}}$$

Importance of decays









Thank you!

All kinds of feedback regarding this course is welcome: radoslaw.ryblewski@ifj.edu.pl

Self-study for this lecture:

R. Ryblewski, *Monte-Carlo statistical hadronization in relativistic heavy-ion collisions*, arXiv:1712.05213