A review of information paradox

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References:

Papers

1) S. Hawking 1975-76.
2) L. Susskind and J. Lindesay, An introduction to black holes and information….
3) D. Harlow, 1409.1231
4) P. Hayden and J. Preskill, arxiv:0708.4025.
5) Susskind,… hep-th/9308100, 9306069, 0808.2096, …

Lectures:

1) Black holes and holography. L. Susskind
2) Jerusalem lectures on black hole and quantum information. D. Harlow.
3) ICTP spring school 2019. K. Papadodimas.

Classical and Quantum Information

Classical information

Liouville's theorem

The Liouville's theorem describes the time evolution of the *phase space.*
Consider an ensemble of many identical states with different initial conditions, then the density of states is constant along every trajectory in phase space.

$$\rho = \rho(p_1, q_1, \cdots, p_N, q_N; t), \qquad \frac{d\rho}{dt} = 0.$$

- The Hamiltonian is allowed to vary with time.
- There are no restrictions regarding how strongly the degrees of freedom are coupled.

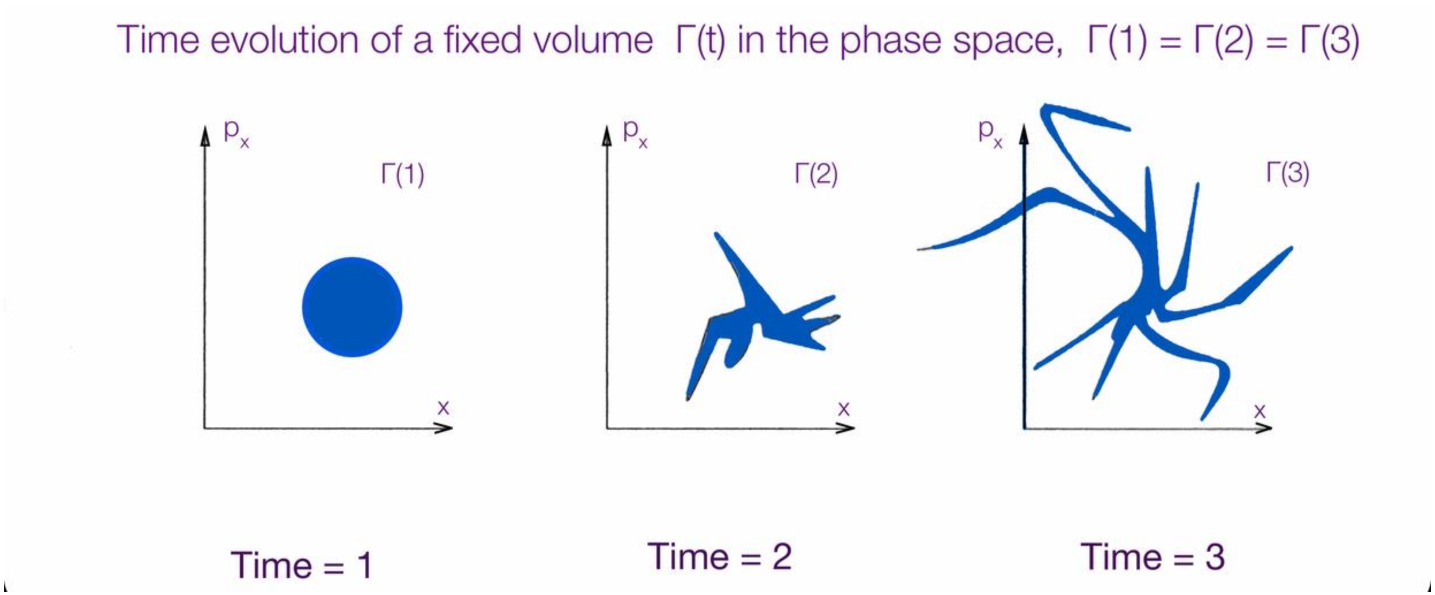
For a region Γ in phase space with probability density $\rho(p,q)$, the volume of the phase space can be define through

$$V_\Gamma \to \exp[S]$$

$$S = -\int dpdq\, \rho(p,q) \log \rho(p,q)$$

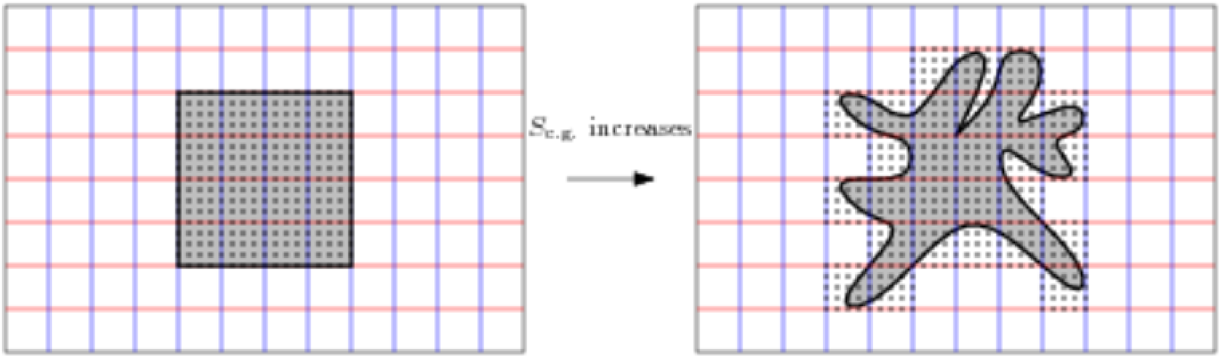
S is called the *fine grained entropy*.

By Liouville's theorem, $\frac{d\rho}{dt} = 0$ and we find that S remains constant in time, $\frac{dS}{dt} = 0$.



For example if $\rho(p,q) = 1/V_\Gamma$ inside the blue region and zero outside, then $S = \log\,[V_\Gamma]$

A fine-grained distribution ρ can be coarse-grained by performing a local average over each cell (or partition) in phase space.



Coarse grained entropy $\bar{S}$ is defined as

$$\bar{S} = -\int \mathrm{dp}\,\mathrm{dq}\,\bar{\rho}(\mathrm{p},\mathrm{q}) \log \bar{\rho}(\mathrm{p},\mathrm{q})$$

where $\bar{\rho}$ is the average density. In fact $\bar{S}(0) < \bar{S}(t)$

This is the origin of the second law of thermodynamics.

The macroscopic properties of a fine grained distribution ρ are completely encoded in its coarse grained version $\bar{\rho}$.

Using the Jensen inequality $S \leq \bar{S}$

Conclusion:
The fine grained information is conserved but the coarse grained information is not.

Quantum information

Similarly for Quantum Mechanics the fine grained or *von Neumann entropy* is

$$S_{vN} = -Tr[\rho \log \rho]$$

where ρ is density matrix of states.

For example for a mixed state

$$\rho_M = \sum_{i=1}^{N} p_i \, |\psi_i\rangle\langle\psi_i| \, , \qquad \sum_{i=1}^{N} p_i = 1$$

$$S_{vN} = -\sum_{i=1}^{N} p_i \log p_i$$

- A pure state yields zero entropy since

$$p_1 = 1 \;\&\; \; p_i = 0, \qquad i \neq 1 \quad \rightarrow \quad S = 0$$

there is no ignorance in knowing the state of the system.

- In contrast to a pure state, for a maximally mixed state

$$p_i = \frac{1}{N} \quad \rightarrow \quad S = \log N$$

Reduced density matrix

Consider a quantum system with only two different degrees of freedom. Its state vector, using the Schmidt decomposition can be written as

$$|\psi\rangle = \sum_{i,j=1}^{N} a_{ij} \, |\chi_i^I\rangle \, |\chi_i^{II}\rangle = \sum_{i=1}^{N} c_i \, |\psi_i^I\rangle \, |\psi_i^{II}\rangle$$

the pure density matrix for the system is given by

$$\rho = |\psi\rangle\langle\psi| = \sum_{i,j=1}^{N} c_i \, c_j^* \, |\psi_i^I\rangle\langle\psi_i^I| \, |\psi_i^{II}\rangle\langle\psi_i^{II}|$$

The reduced density matrix is defined by

$$\rho_R^I = Tr_{II}[\rho] = \sum_{i=1}^{N} |c_i|^2 \, |\psi_i^I\rangle\langle\psi_i^I|$$

- Performing a partial trace erases information about the degree of freedom, and hence, the density matrix of a mixed state contains less information than a pure state.
- The density matrix for a mixed state is a precise measure of how much information is lost in performing an observation on a quantum system.

Properties

1. $Tr[\rho_R] = 1.$

2. $\rho_R = {\rho_R}^\dagger$ (positive semidefinite, i.e. all eigen-values ≥ 0) .

3. If $Tr[\rho_{I \oplus II}]$=0 (the whole system in pure state) then

non-zero eigen-values of I = non-zero eigen-values of II.

Therefore if $S_{I \oplus II} = 0$ then $S[\rho_R^I] = S[\rho_R^{II}]$.

Entangled Quantum States

If the joint state vectors cannot be factorized, which are called *entangled states*, the two degrees of freedom become *inseparable*, and one cannot consider either of the degrees of freedom independently of the other.
For instance in the previous example

$$\rho_R^I = Tr_{II}[\rho] = \sum_{i=1}^{N} |c_i|^2 \, |\psi_i^I\rangle\langle\psi_i^I|$$

$$Tr[\rho_R^I] = \sum_{i=1}^{N} |c_i|^2 = 1\,, \qquad Tr\left[{\rho_R^I}^2\right] = \sum_{i=1}^{N} |c_i|^4 < 1$$

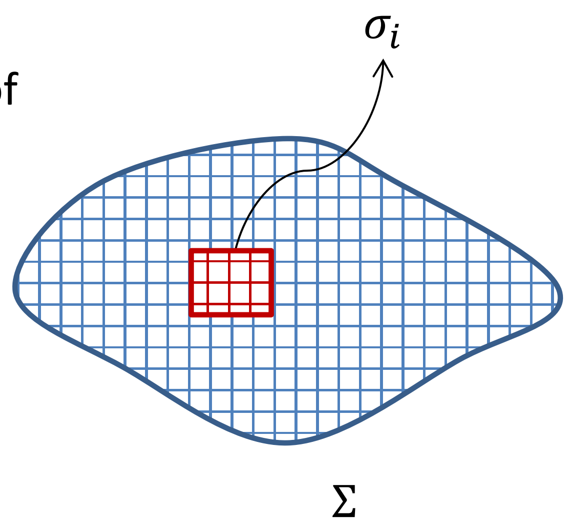
The last inequality leads to the conclusion that in any set of basis

$$|\psi\rangle = \sum_{i=1}^{N} c_i \, |\psi_i^I\rangle \, |\psi_i^{II}\rangle \neq |\chi^I\rangle|\chi^{II}\rangle$$

The Entanglement Entropy is

$$S_{EE} = -Tr[\rho_R \log \rho_R]$$

Fine grained and coarse grain entropy in quantum systems



Consider a pure system Σ with total energy E which is a union of weakly interacting thermal similar subsystems σ_i

$$\Sigma = \bigcup_{i=1}^{N} \sigma_i$$

The density matrix of each subsystem is given by (a general property of most complex interacting systems)

$$\rho_i = e^{\beta H_i}/Z_i$$

where H_i is the energy of each subsystem, (The thermal density matrix maximizes the entropy for a given average energy).
Then the coarse grained (thermal) entropy can be defined as

$$S_{Theraml} = \sum_{i=1}^{N} S_i = -\sum_{i=1}^{N} \rho_i \log \rho_i$$

The fine grained entropy is (Σ is pure)

$$S_{F.G.} = 0$$

The coarse grained entropy is what we usually think of in the context of thermodynamics. It is not conserved:

For example suppose we start with the subsystems in a product state with no correlations

$$\psi_\Sigma = \prod_{i=1}^{N} \psi_{\sigma_i}$$

Therefore

$$S_i = 0 \quad \rightarrow \quad S_{Theraml} = \sum_{i=1}^{N} S_i = 0$$

and fine grained entropy (ψ_Σ is pure)

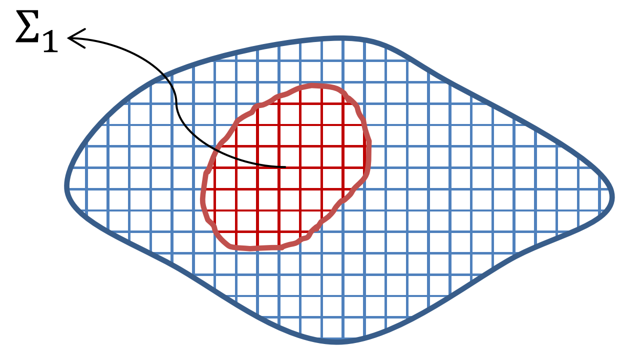
$$S_\Sigma = 0,$$

Now subsystems interact and ψ_Σ is not factorizable, therefore

$$S_i \neq 0 \quad \rightarrow \quad S_{Thermal} \neq 0$$

But $S_\Sigma = 0$ because the fine grained entropy is conserved.

In other words by a unitary time evolution $\psi_\Sigma^{pure}[t] = U(t, t_0)\psi_\Sigma^{pure}[t_0]$.



Now consider $\Sigma_1 \in \Sigma$

Denote the fine grained entropy of Σ_1 with the remaining subsystem $\Sigma - \Sigma_1$ as $S[\Sigma_1]$.
This can be computed from the entanglement entropy of the reduced density matrix $\rho_R[\Sigma_1]$.

$$S_{EE}[\Sigma_1] = S_{F.G.}[\Sigma_1] \equiv S[\Sigma_1] = -Tr[\rho_R[\Sigma_1] \log \rho_R[\Sigma_1]]$$

Always we have

$$S[\Sigma_1] \leq S_{thermal}[\Sigma_1]$$

For example

If $\quad \Sigma_1 = \Sigma \rightarrow S[\Sigma] = 0 \leq S_{thermal}[\Sigma_1]$

or if $\quad \Sigma_1 \ll \Sigma \rightarrow S\left[\Sigma_1\right] = S\left[\Sigma - \Sigma_1\right] \approx S[\Sigma] = 0 \leq S_{thermal}[\Sigma_1]$

Definition of Information: Information is the difference between coarse grained and fine grained entropy

$$I = S_{Thermal} - S_{F.G.}$$

Two extremes

1) For a small subsystem ($\bar{\rho} = \rho$) $S_{F.G.} \approx S_{C.G} = S_{Thermal} \rightarrow \ I \approx 0.$
2) For pure system Σ : $S_{F.G.} = 0 \ \rightarrow \ I = S_{Thermal}.$

This information is hidden in correlation between subsystems which make Σ pure.

Theorems on quantum information of subsystems

- How much information are in a moderately sized subsystem?
- Don Page: For subsystems smaller than about the half of the total system, $I \approx 0$.

Results:

If $\Sigma_1 < \frac{1}{2}\Sigma$ then $I[\Sigma_1] \approx 0 \quad \rightarrow \quad S[\Sigma_1] \cong S_{Thermal}[\Sigma_1]$

Σ is pure so $\quad S[\Sigma - \Sigma_1] = \mathrm{S}[\Sigma_1]$

If $\Sigma_1 > \frac{1}{2}\Sigma \rightarrow \Sigma - \Sigma_1 < \frac{1}{2}\Sigma$

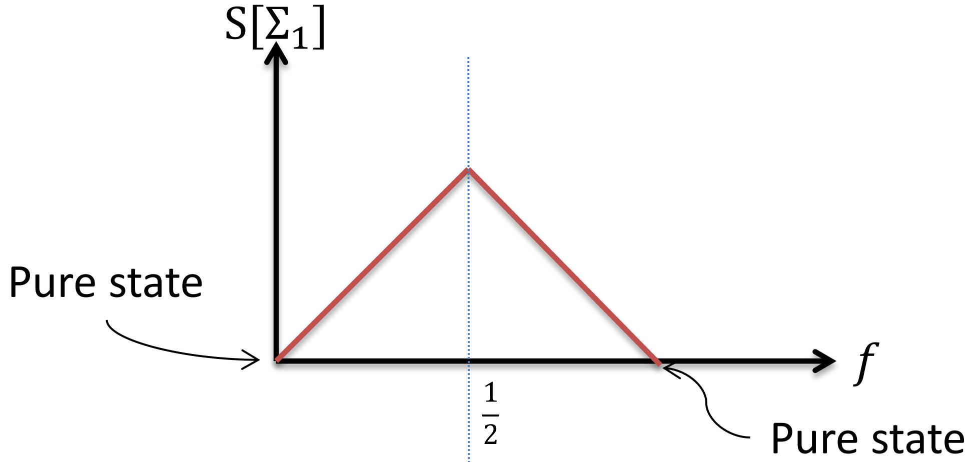
$S[\Sigma - \Sigma_1] \cong S_{Thermal}[\Sigma - \Sigma_1] \rightarrow S[\Sigma_1] \cong S_{Thermal}[\Sigma - \Sigma_1]$

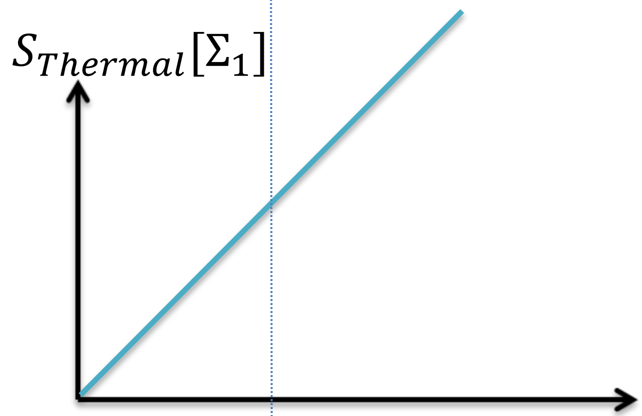
$$S_{Thermal}[\Sigma - \Sigma_1] \cong (1 - f) S_{Thermal}[\Sigma]$$

$$f = \frac{\#d.o.f\ \Sigma_1}{\#d.o.f\ \Sigma}$$

$$I[\Sigma_1] = \mathrm{S_{Thermal}}[\Sigma_1] - \mathrm{S}[\Sigma_1] = (2\mathrm{f} - 1)\ \mathrm{S_{Thermal}}[\Sigma]$$

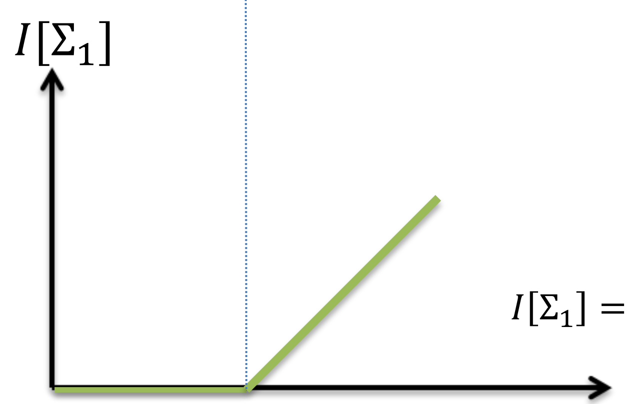
Page Curve





$$S_{Thermal}[\Sigma_1] \cong f\ S_{Thermal}[\Sigma]$$

$$f = \frac{\#d.o.f\ \Sigma_1}{\#d.o.f\ \Sigma}$$



$$I[\Sigma_1] = \mathrm{S}_{\mathrm{Thermal}}[\Sigma_1] - \mathrm{S}[\Sigma_1] = (2\mathrm{f} - 1)\ \mathrm{S}_{\mathrm{Thermal}}[\Sigma]$$

$$f \geq 1/2$$

- In a large quantum system most pure states looks almost identical to the maximally mixed state when probed by an observable.

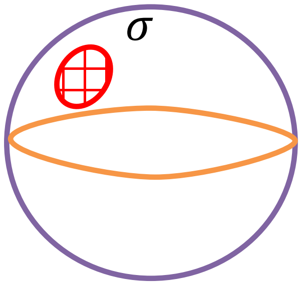
Consider a finite dimensional Hilbert space with $\dim H = N \gg 1$.
$|i\rangle$ = orthonormal basis of H $\quad i = 1,2, \ldots, N$.
A = A linear observable acting on H.

The most general pure state in H is denoted by

$$|\psi\rangle = \sum_{i=1}^{N} c_i \, |i\rangle, \qquad \sum_{i=1}^{N} |c_i|^2 = 1,$$

So the set of pure states lives on a $2N - 1$ dimensional unit sphere.

Consider each pure state is equally likely (microcanonical or Haar measure).
The probability to select states inside $\sigma \sim V_\sigma$



Haar measure:

$$d\mu = a \, dc_1 dc_1^* \ldots dc_N dc_N^* \, \delta(\sum_{i=1}^{N} |c_i|^2 = 1) \; , \qquad with \qquad \int d\mu = 1$$

The average of $\langle A \rangle$ over all pure state is

$$\overline{\langle\psi|A|\psi\rangle} = \int d\mu \langle\psi|A|\psi\rangle = \sum_{i,j} A_{ij} \int d\mu\, c_i\, c_j^* = \frac{1}{N}\, A_{ii} = Tr[\rho_m A]$$

where

$\rho_m = \frac{1}{N}$ = micro-canonical, maximally mixed state density matrix.

The standard deviation is

$$\overline{(\langle\psi|A|\psi\rangle - Tr[\rho_m A])^2} = \frac{1}{N+1}(Tr[\rho_m A^2] - Tr[\rho_m A]^2)$$

At large $N \gg 1$ the right hand side tends to zero, so

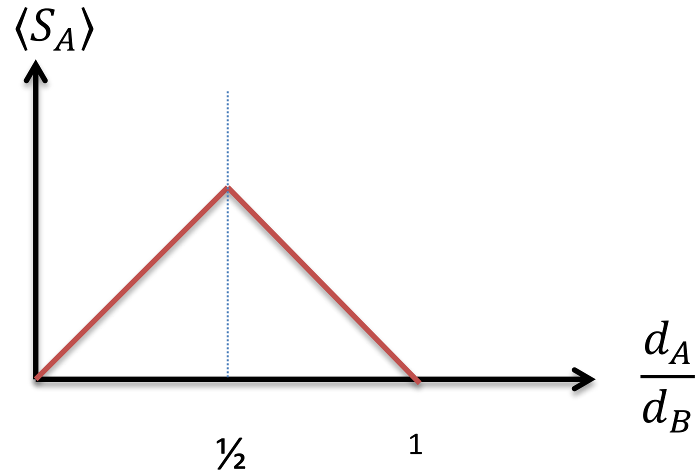
$$\langle\psi|A|\psi\rangle = Tr[\rho_m A] + \mathcal{O}(\frac{1}{N})$$

So most pure states are identical to each other and also close to maximally mixed state.

For example consider a quantum system with a bipartite Hilbert space $H = H_A \otimes H_B$ where the subsystems A and B have dimension $d_A = \dim H_A$, $d_B = \dim H_B$, with A the smaller of the two subsystems, $1 < d_A \leq d_B$. Given a pure state $|\psi\rangle \in H$, the subsystems A and B have the same entanglement entropy $S_A = S_B$. For $d_B \gg 1$

$$\langle S_A \rangle = -\langle Tr(\rho_A \log \rho_A)\rangle = -\lim_{r\to 1} \partial_r \langle Tr \rho_A^r \rangle \approx \log d_A - \frac{1}{d_A d_B} \frac{d_A^2 - 1}{2}$$

$$(\Delta S_A)^2 = \langle S_A^2 \rangle - \langle S_A \rangle^2 \approx \frac{1}{d_A d_B} \sqrt{\frac{d_A^2 - 1}{2}}$$



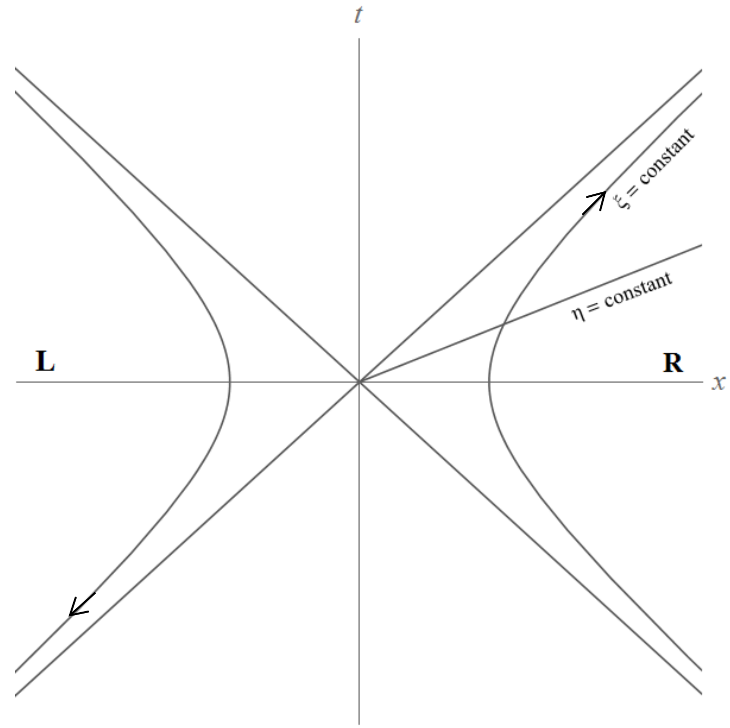
If $d_A < \frac{1}{2} d_B$ then ρ_A is exponentially close to maximally mixed state and its entanglement Entropy is $S_A = \log d_A$. If $d_A > \frac{1}{2} d_B$ then B will be the small subsystem.

Accelerating observer in flat space

Flat space-time: Minkowski vs Rindler coordinates
$ds^2 = -dt^2 + dx^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$

Rindler coordinates describe a uniformly accelerating observer.

The Rindler horizon is the boundary of the Rindler coordinates. The Rindler observer can’t receive any signal from $t \geq x$ or send signal to $t \leq -x$. So the Rindler space is divided into two distinct Right and Left wedges.



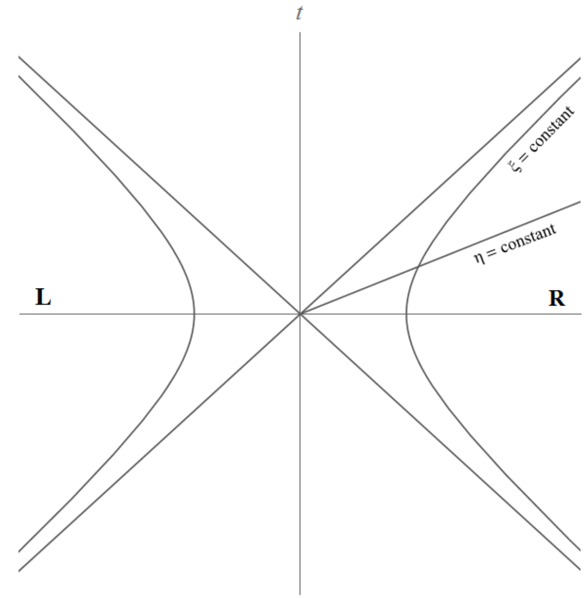
$\mathcal{M}$: Time translation symmetry; $i\partial_t \to H$
$H =$ Minkowski Hamiltonian.

$\mathfrak{R}$: Rindler time translation symmetry; $i\partial_\eta \to K$
$K =$ Boost (Rindler Hamiltonian)

QFT on Rindler space

Consider a scalar field theory on a two dimensional Minkowski space-time

$$S = \int d^2x \sqrt{-g}\, (\nabla\phi)^2$$



Positive energy solutions

$\mathcal{M}$: $$f_k \sim e^{ikx - i\omega t}\,, \quad |k| = \omega$$

$\mathfrak{R}$: $$g_q^{L/R} \sim e^{iq\xi \pm i\nu\eta}\, \Theta(\mp x)\,, \ |q| = \nu$$

Quantization

$$\phi_{\mathcal{M}} = \sum_k a_k\, f_k + a_k^\dagger\, f_k^*$$

$$\phi_{\mathfrak{R}} = \sum_q \tilde{b}_q g_q^L + \tilde{b}_q^\dagger g_q^{L^*} + b_q g_q^R + b_q^\dagger g_q^{R^*}$$

A basis change between a and $b, \tilde{b}$

$$b_q = \alpha_{qk}\, a_k + \beta_{qk}\, a_k^\dagger$$ Bogoliubov coefficients

States:

$$H_{QFT} = H_{\mathcal{M}} = H_L \otimes H_R$$

$$|N\rangle_{\mathcal{M}} \in H_{\mathcal{M}},\ |n_L\rangle|m_R\rangle \in H_L \otimes H_R$$

Minkowski vacuum state:

$$|0\rangle_{\mathcal{M}}: \qquad a_k|0\rangle_{\mathcal{M}} = 0$$

$$b_q|0\rangle_{\mathcal{M}} = \beta_{qk} a_k^\dagger |0\rangle_{\mathcal{M}} \neq 0$$

The notion of vacuum is not invariant with respect to which observer we consider.

What does $|0\rangle_{\mathcal{M}}$ look like in terms of Rindler states?

There is a basis of a_k that $$a_k \sim b_k - e^{-\frac{i\pi}{a}\nu}\, \tilde{b}_{-k}$$

where

$$|0\rangle_{\mathcal{M}} = \prod_k \sum_n e^{-\frac{\pi}{a}n\nu} |n_{-k}\rangle_L |n_k\rangle_R$$

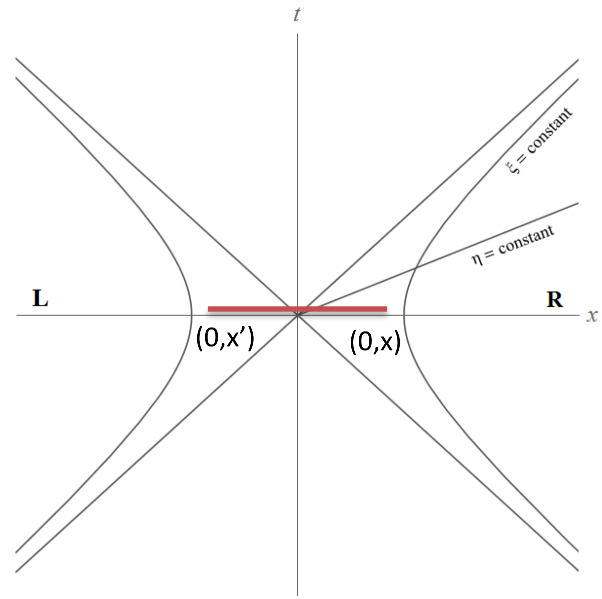
(Thermofield double state)

It’s an entangled state between modes on the left and modes on the right of Rindler space.

The reduced density matrix is

$$\rho_R \sim e^{-\frac{2\pi}{a}K}$$

So an observer who is constantly accelerating in Minkowski space, experiences a thermal bath at $T = \frac{a}{2\pi}$ (Unruh radiation). The proper temperature is given in terms of the proper acceleration $T_{obs} = \frac{\alpha}{2\pi} = \frac{ae^{-a\xi}}{2\pi}$.



Entanglement

Consider two following density states

$\rho_M^0 = |0\rangle_M\langle 0|$, $\quad$ *Entangled*

$\rho_M = \rho_L \otimes \rho_R$, $\quad$ Unentangled

By point splitting

$$T_{tt} = \lim_{\substack{t'\to t \\ x'\to x}} \partial_t\phi(t,x)\partial_{t'}\phi(t',x') - {}_M\langle 0|\, \partial_t\phi(t,x)\partial_{t'}\phi(t',x')|0\rangle_M$$

At origin

$$\langle T_{tt}\rangle = Tr[\rho_M^0 T_{tt}(0)] = 0, \qquad \langle T_{tt}\rangle = Tr[\rho_M\, T_{tt}(0)] \to \infty.$$

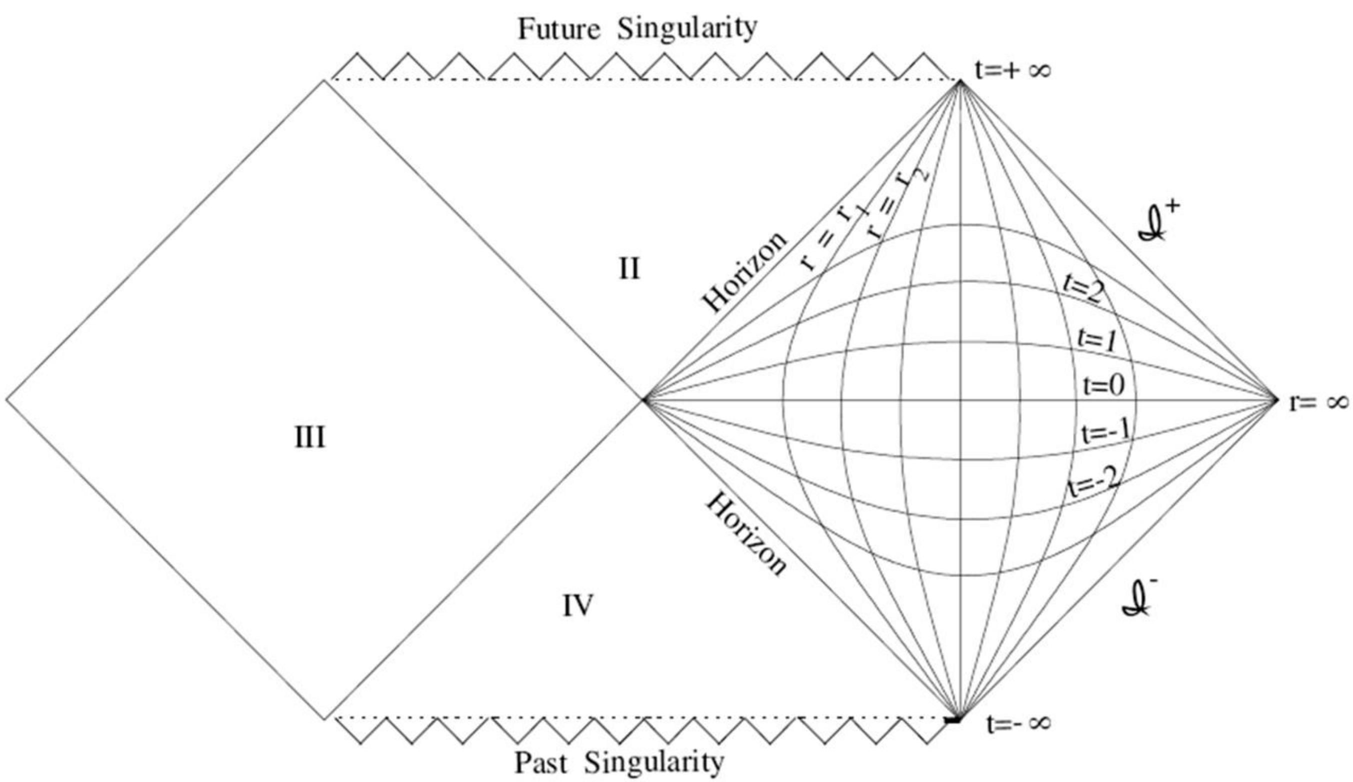
Black holes

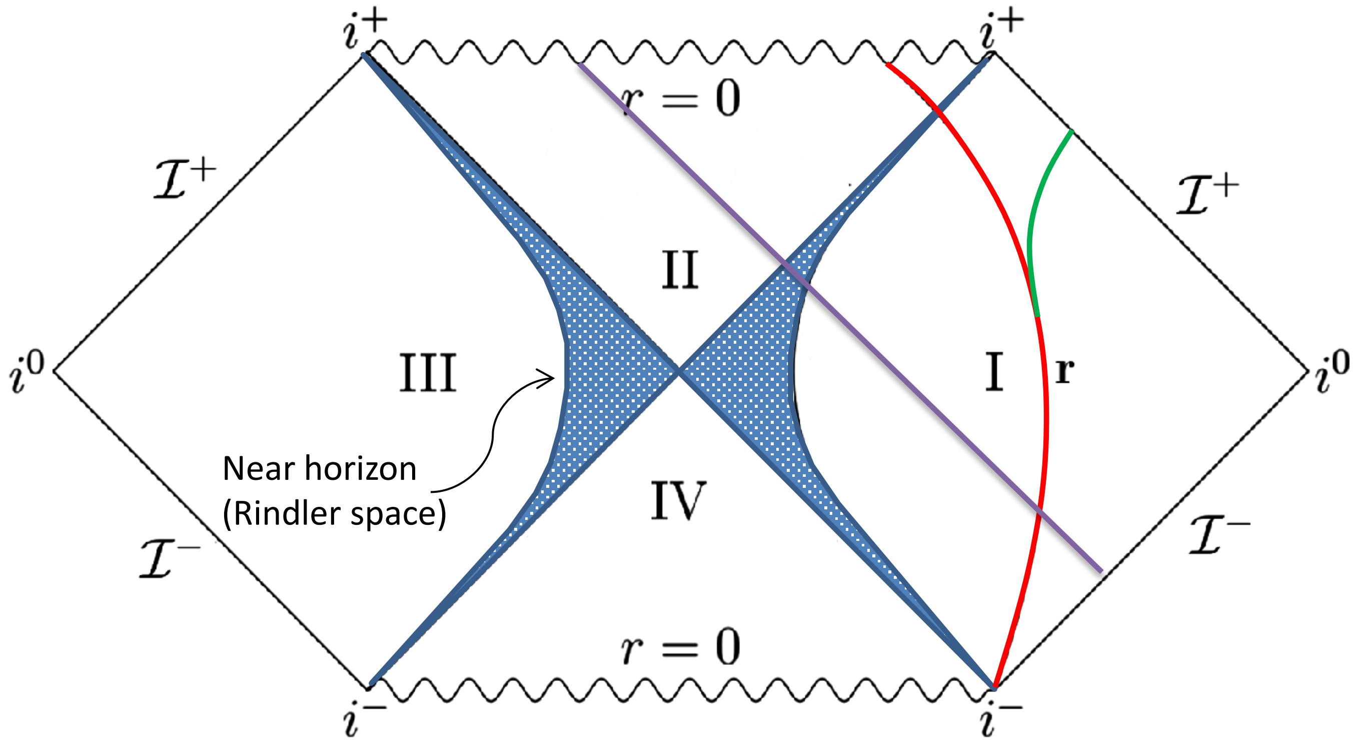
Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2$$

Eternal black hole

Penrose diagram for Schwarzschild black hole





$$\rho = \int_{r=2MG}^{r} \sqrt{g_{rr}} dr \ , \qquad x = 2MG\ \theta \cos\phi \ , \qquad y = 2MG\ \theta \sin\phi \ , \qquad \omega = \frac{t}{4MG}$$

Near horizon is similar to Rindler space

$$ds^2 = -\rho^2\, d\omega^2 + d\rho^2 + dx^2 + \mathrm{dy}^2$$

Moreover

$T = \rho \sinh\omega \ , z = \rho\cosh\omega \ \rightarrow \ ds^2 = -dT^2 + dx^2 + dy^2 + dz^2$

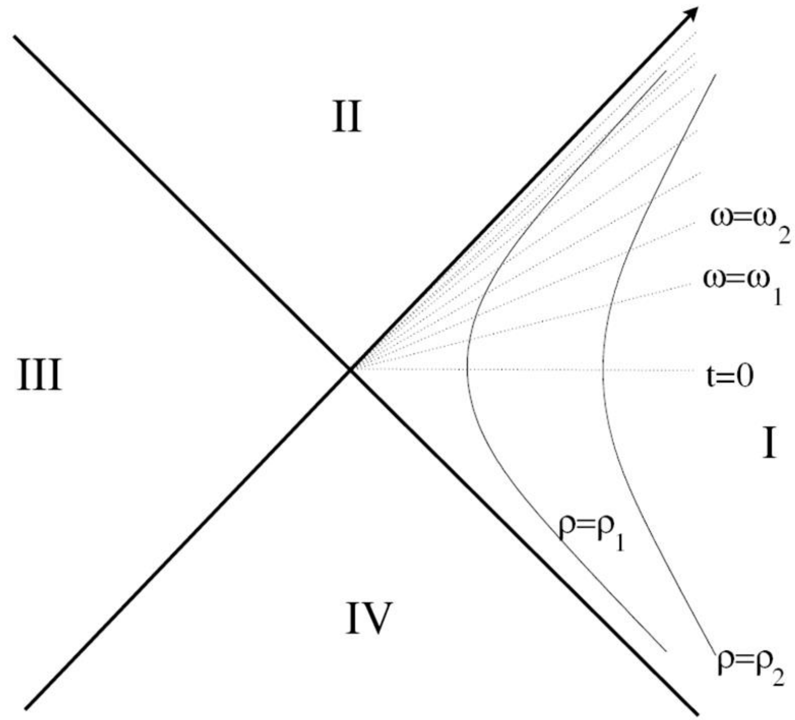
A large black hole is locally almost indistinguishable from flat space-time.

- The same entanglement exist between modes on the left and right Rindler wedges.
- The vacuum is similar to $|0\rangle_{\mathcal{M}} = |0\rangle_{HH}$ (Hartle-Hawking vacuum state).
- Black hole is in thermal equilibrium with radiation coming from infinity at temperature

$$T_H = \frac{\hbar c^3}{8\pi k_B} \frac{1}{GM}$$

- Thermodynamics

$$dE = T_H dS \rightarrow dS = 8\pi GM\, dM \rightarrow S = 4\pi GM^2$$
$$A_H = 4\pi r_H^2 = 4\pi (2GM)^2 = 16\pi G^2 M^2$$
$$\rightarrow\ S_{BH} = \frac{A_H}{4G}$$



The ground state is given by

$$\Psi(\chi_L, \chi_R) = \frac{1}{\sqrt{Z}} \int_{T>0} d\chi(x) e^{-I_E}$$

By going to Rindler coordinates

$$I_E \rightarrow H_{\Re} = K$$

boost (generator of ω translation)

$$H_{\Re} = \int_{\rho=0}^{\infty} d\rho\, dx\, dy\, \rho\, T^{\omega\omega}$$

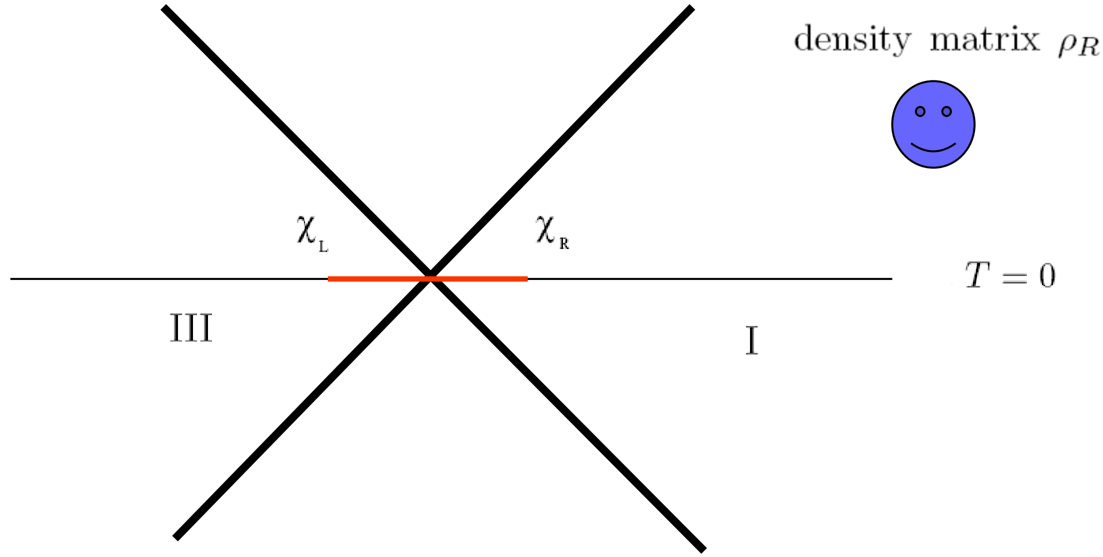
For example

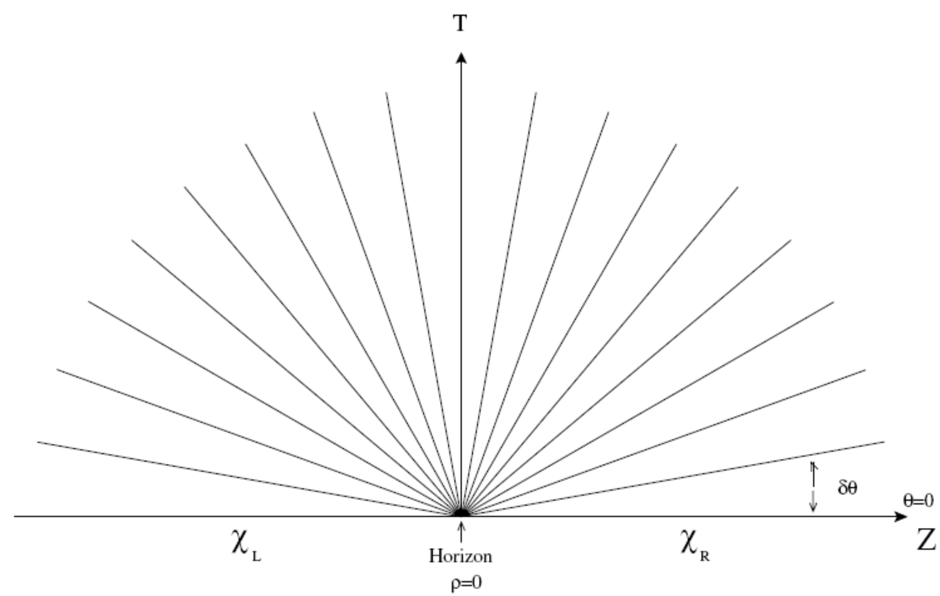
$$T^{\omega\omega} = \frac{\Pi^2}{2} + \frac{1}{2}(\nabla\chi)^2 + V(\chi)$$

So the ground state in Rindler coordinate is written as

$$\Psi(\chi_L, \chi_R) = \frac{1}{\sqrt{Z}} \langle \chi_L | e^{-\pi K} | \chi_R \rangle$$

The ground state is a transition matrix element between initial state $|\chi_R\rangle$ and final state $|\chi_L\rangle$.





$$\rho_R(\chi_R, \chi_R') = \int \Psi^*(\chi_L, \chi_R)\Psi(\chi_L, \chi_R')\, d\chi_L = \frac{1}{Z}\int \langle \chi_R | e^{-\pi K} | \chi_L\rangle \langle \chi_L | e^{-\pi K} | \chi_R'\rangle\, d\chi_L$$

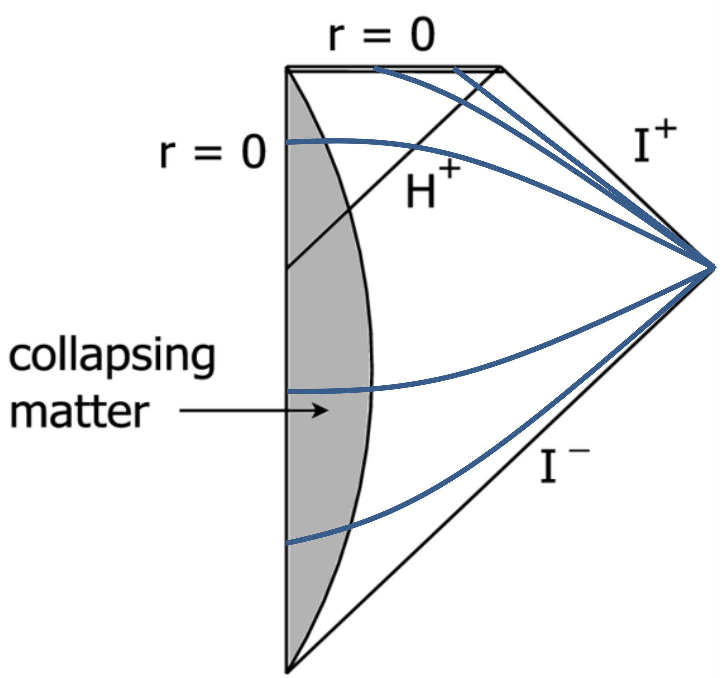
$$= \frac{1}{Z}\langle \chi_R | e^{-2\pi K} | \chi_R'\rangle$$

So the observer on the right Rindler wedge see the vacuum as a thermal ensemble with density matrix given by

$$\rho_R = \frac{1}{Z} e^{-2\pi K}$$

The temperature of the ensemble is $T_R = \frac{1}{2\pi}$ and the proper temperature is $T(\rho) = \frac{1}{2\pi\rho}$.

Collapsing black hole



Penrose diagram of collapsing BH = Minkowski + Schwarzschild

Quantum properties of the scalar fields in the curved background of black holes

Tortoise coordinates

$$r^* = r + 2MG \log(\frac{r}{2MG} - 1)$$

$$ds^2 = \left(1 - \frac{2MG}{r}\right)(-dt^2 + dr^{*2}) + r^2 d\Omega^2$$

This coordinate covers only $r > 2MG$ and horizon is located at $r^* \to -\infty$.

Starting from massless scalar field

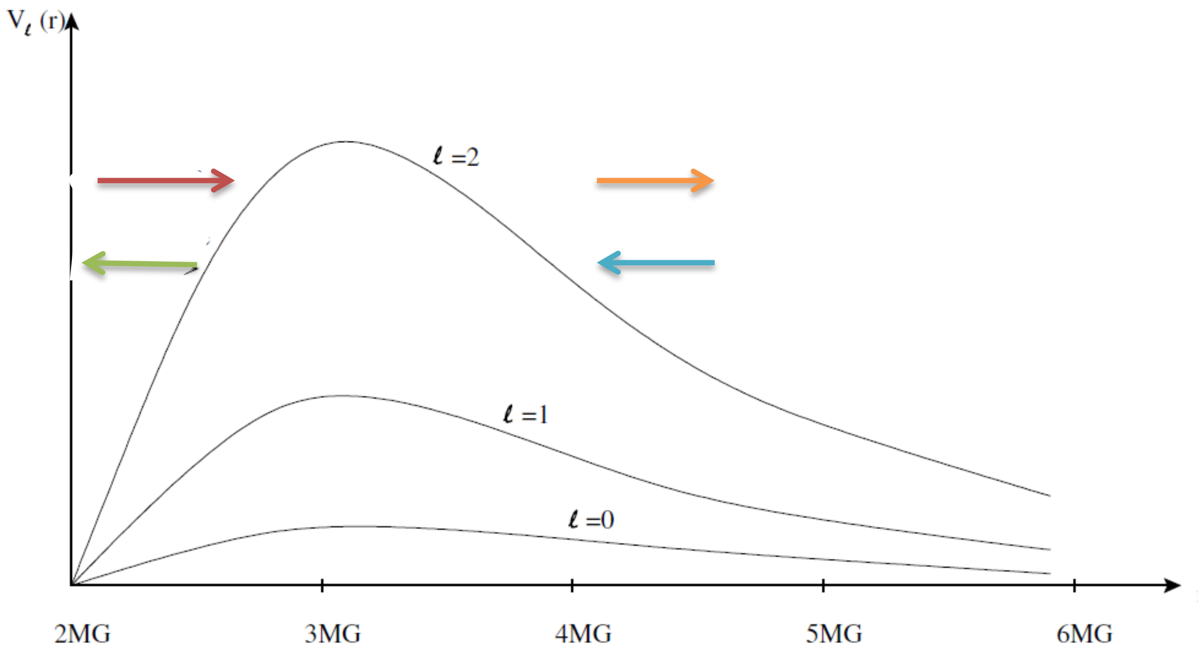
$$I = \int d^4x \sqrt{-g} \frac{1}{2} (\nabla\chi)^2$$

and by defining $\psi = r\chi$

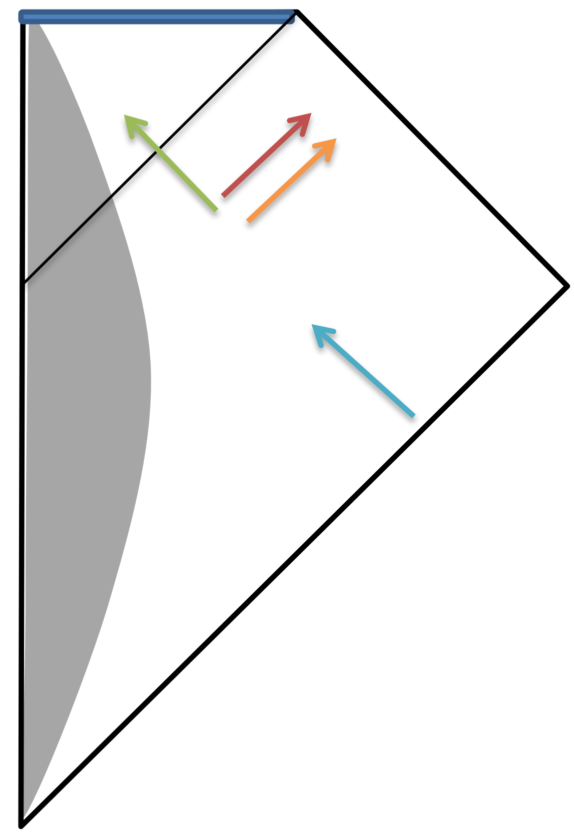
$$I_{lm} = \frac{1}{2} \int dt\, dr^* \, [-\left(\frac{\partial \psi_{lm}}{\partial t}\right)^2 + \left(\frac{\partial \psi_{lm}}{\partial r^*}\right)^2 + \mathrm{V}_l(\mathrm{r}^*)\psi_{lm}^2]$$

For a mode of frequency ν, the equation of motion is a Schrodinger equation with energy $\nu^2$and potential as

$$\mathrm{V}_l(\mathrm{r}^*) = \left(1 - \frac{2\mathrm{MG}}{\mathrm{r}}\right)\left(\frac{\mathrm{l(l+1)}}{\mathrm{r}^2} + \frac{2\mathrm{GM}}{\mathrm{r}^3}\right)$$



V_ℓ (r)
ℓ =2
ℓ =1
ℓ =0
r
2MG
3MG
4MG
5MG
6MG



Quantization in curved space-time

Let's suppose that there is no back-reaction of quantum fields on classical metric.

Consider a massless Klein-Gordon field

$$\Box\phi = 0$$

On Cauchy slice I^-

$$\phi(x) = \sum_i f_i\, a_i + \bar{f}_i\, a_i{}^\dagger$$

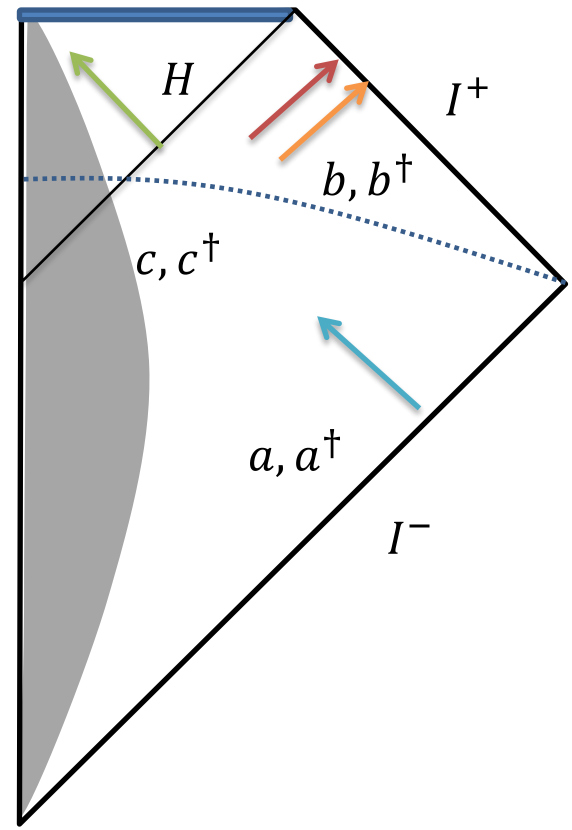
$$\left[a_{\omega l m}\ , a^\dagger_{\omega' l' m'}\right] = \delta(\omega' - \omega)\delta_{ll'}\delta_{mm'}$$

On Cauchy slices $I^+ \cup H$

$$\phi(x) = \sum_i g_i\, b_i + \bar{g}_i\, b_i{}^\dagger + h_i c_i + \bar{h}_i\, c_i{}^\dagger$$

where $\{f_i, \bar{f}_i\}$ are complete basis on I^- and $\{g_i, \bar{g}_i, h_i\,, \bar{h}_i\}$ are complete basis on $I^+ \cup H$.

$$\left[b_i, b_j{}^\dagger\right] = \left[c_i, c_j{}^\dagger\right] = \delta_{ij}, \qquad \left[b_i, c_j\right] = 0.$$



Transformation between two basis

$$b_i = \sum_j \alpha_{ij}\, a_j + \beta_{ij} a_j^\dagger$$

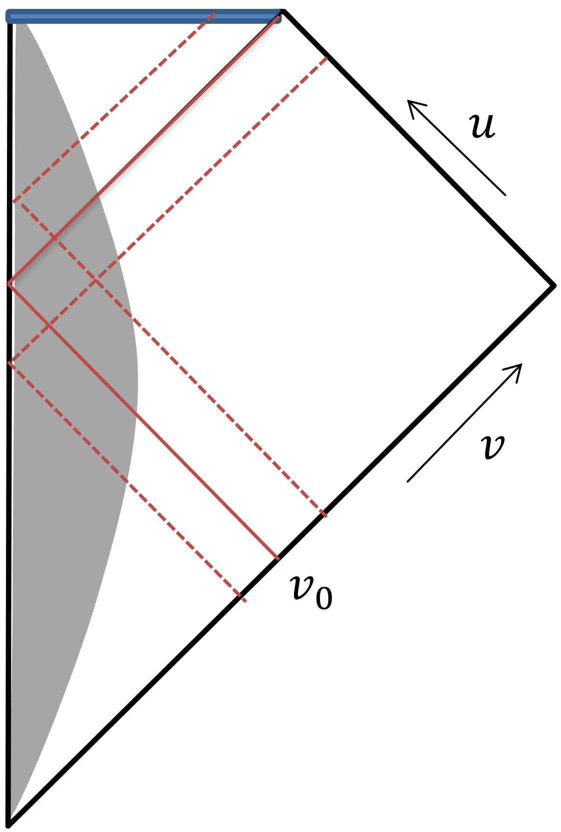
if $a_i|0\rangle_{I^-} = 0$ then ${}_{I^-}\langle 0\left|b^\dagger b\right|0\rangle_{I^-} = \sum\left|\beta_{ij}\right|^2 \neq 0.$

- This means that if we start from a vacuum state in the far past, we will get a flux of particles in the far future.
- This is because that the collapsing black hole has a time dependent geometry, so the expansion in the early times is different from expansion at late times.

What is the outgoing particle distribution?

To find β_{ij} we need to solve the wave equation of motion in I^+ and I^- and relate them.

Hawking approach to estimate the results



Tortoise coordinates:

$$r^* = r + 2MG\log(\frac{r}{2MG} - 1)$$

$$v = t + r^*, \qquad u = t - r^*,$$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du\, dv + r^2 d\Omega^2$$

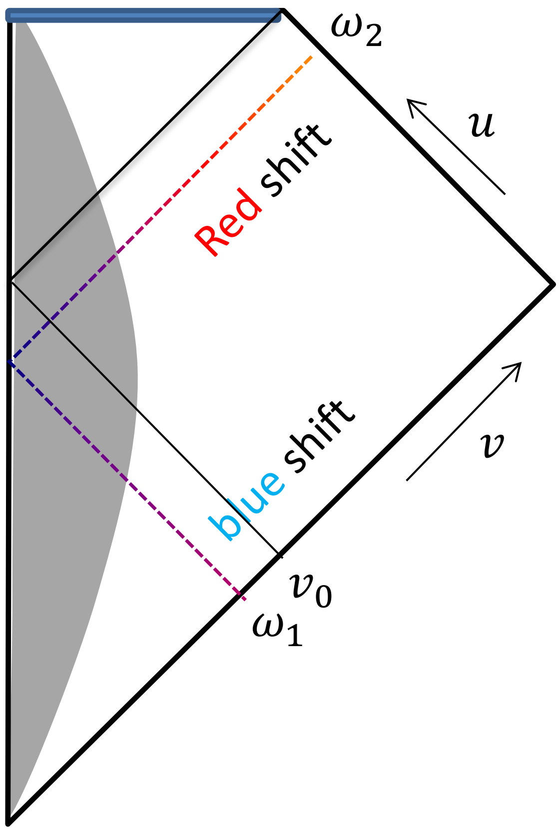
Infalling light rays: $v = const.$
Outgoing light rays: $u = const.$

v_0 is a special light ray in the far past that gets reflected at $r = 0$ and becomes the horizon.

any light ray with $v < v_0$ is going to reach $r = 0$ before the horizon is formed and fly out to infinity.

any light ray with $v > v_0$ is going to reach $r = 0$ after the horizon is formed and will fall into the singularity.

All very late light rays on I^+ are coming from small region very close to $v = v_0$. Small region near v_0 is magnified by black hole into infinite region in u parameter.



For eternal black hole $\omega_1 = \omega_2$.
For collapsing black hole there is a net red shift $\omega_1 > \omega_2$.

UV modes on $I^- \xrightarrow{mapped}$ IR modes on I^+

So we can use the geometric optics approximation without having to solve the wave equations.

Then the only thing we need to know is the relationship between the null geodesics. At late time or large u

$$u \approx -4GM \log(v_0 - v) + const.$$

so we can estimate the Bogoliubov coefficients, α and β.

$$\langle b^{\dagger}_{\omega} b_{\omega'} \rangle = \delta(\omega - \omega') \frac{P(\omega, l)}{e^{\beta\omega} - 1}$$

$P(\omega, l) =$ gray body factor (probability of particles to absorb by black hole).
The outgoing particle distribution describes by a thermal radiation of temperature

$$T = \frac{1}{\beta} = \frac{1}{8\pi GM}$$

So the outgoing modes are thermally populated.

Moreover the modes are uncorrelated, i.e. the higher point functions factorize

$$\langle \overbrace{b \dots b}^{2N} \rangle = \prod_{k=1}^{N} \langle bb \rangle$$

$$\langle \left(b^{\dagger}_{\omega} b_{\omega'}\right)^{k} \rangle = \frac{1}{Z}\, Tr[e^{-\beta\omega b^{\dagger} b} \left(b^{\dagger} b\right)^{k}]$$

$$Z = Tr[e^{-\beta\omega b^{\dagger} b}] = \frac{1}{e^{\beta\omega} - 1}$$

Black hole evaporation

By radiation black hole gets smaller and eventually disappears. The luminosity is given by

$$L = \frac{dM}{dt} = \sum_l \int_0^\infty \frac{d\omega}{2\pi} \frac{\omega\, P(\omega, l)}{e^{\beta\omega} - 1}$$

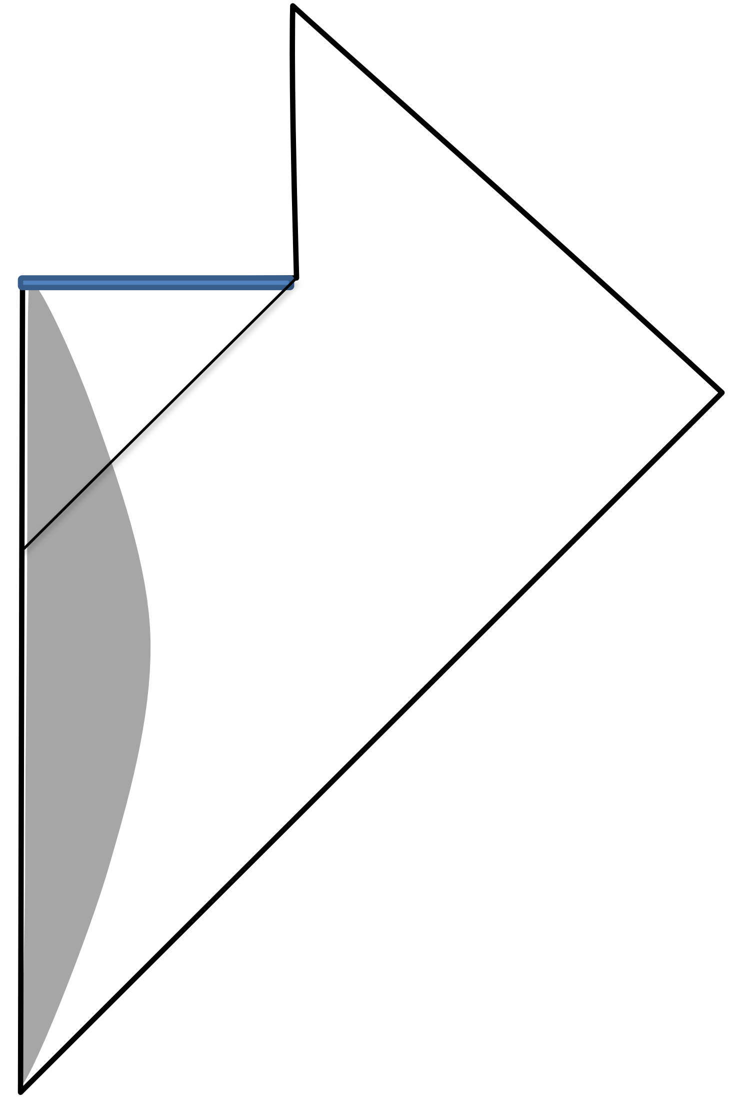
for $l \to 0$: $P(\omega, l) = 4G^2M^2\omega$

$$\frac{dM}{dt} = -\frac{1}{T_H^2} = -\frac{1}{G^2M^2} \to M(t) = M_0\left(1 - \frac{3t}{G^2M_0^3}\right)^{1/3}$$

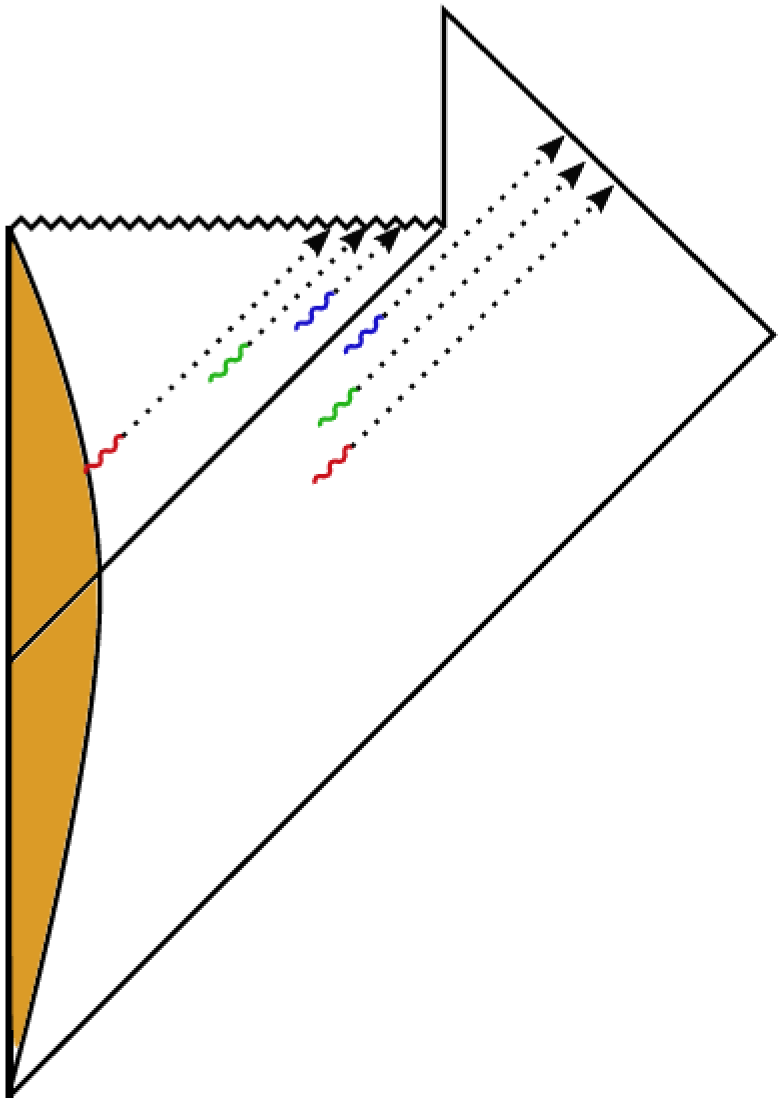
therefore the evaporation time is

$$t_{evap} \sim G^2M_0^3$$

- This approximation is valid until the mass of black hole is of size of the Planck mass.



The Hawking process creates entangled pairs, one trapped behind the horizon and the other escaping to infinity where it is observed as (approximate) blackbody radiation.



Information paradox

Hawking computation seems to contradict unitarity in quantum mechanics.

$$\rho^{I^-} = |shell\rangle\langle shell| \;(pure) \rightarrow \rho^{I^+} = \rho_{Thermal} \;(mixed).$$

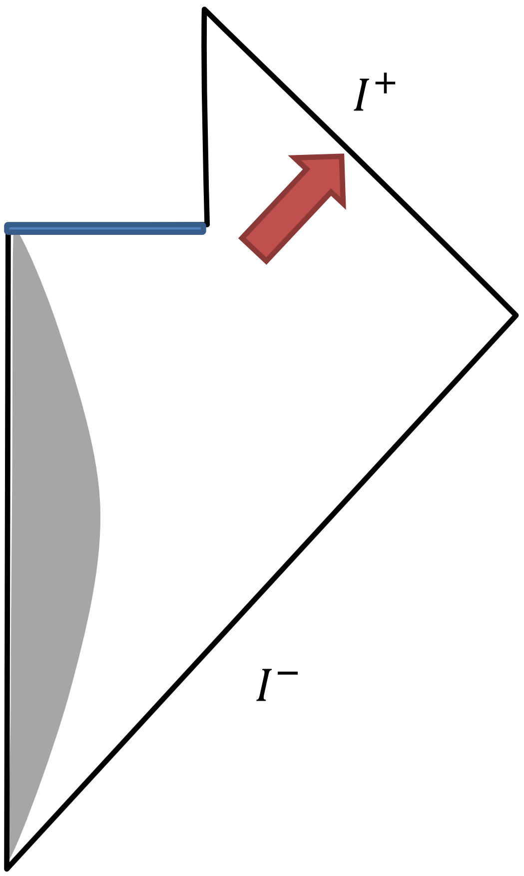
but in quantum mechanics

$$\rho^{I^+} = U\,\rho^{I^-} U^\dagger\,, \qquad U = e^{-iHT}\Big|_{T\rightarrow\infty}$$

therefore

$$\underbrace{S_{vN}[\rho^{I^+}]}_{\neq 0} = \underbrace{S_{vN}[\rho^{I^-}]}_{=0}$$

In other words the radiation only depends on mass of the black hole, so we can make a black hole with a specific mass in different ways and we always get the same final state. By reversing the process we cannot find what was constructed the black hole.



Entropy of Hawking radiation

The von Neumann entropy from the reduced density matrix of the first N hawking particles is

$$S_N = -Tr[\rho_N \log \rho_N]$$

$$N = 1: \qquad S_1 = -Tr[\rho_1 \log \rho_1] \neq 0.$$

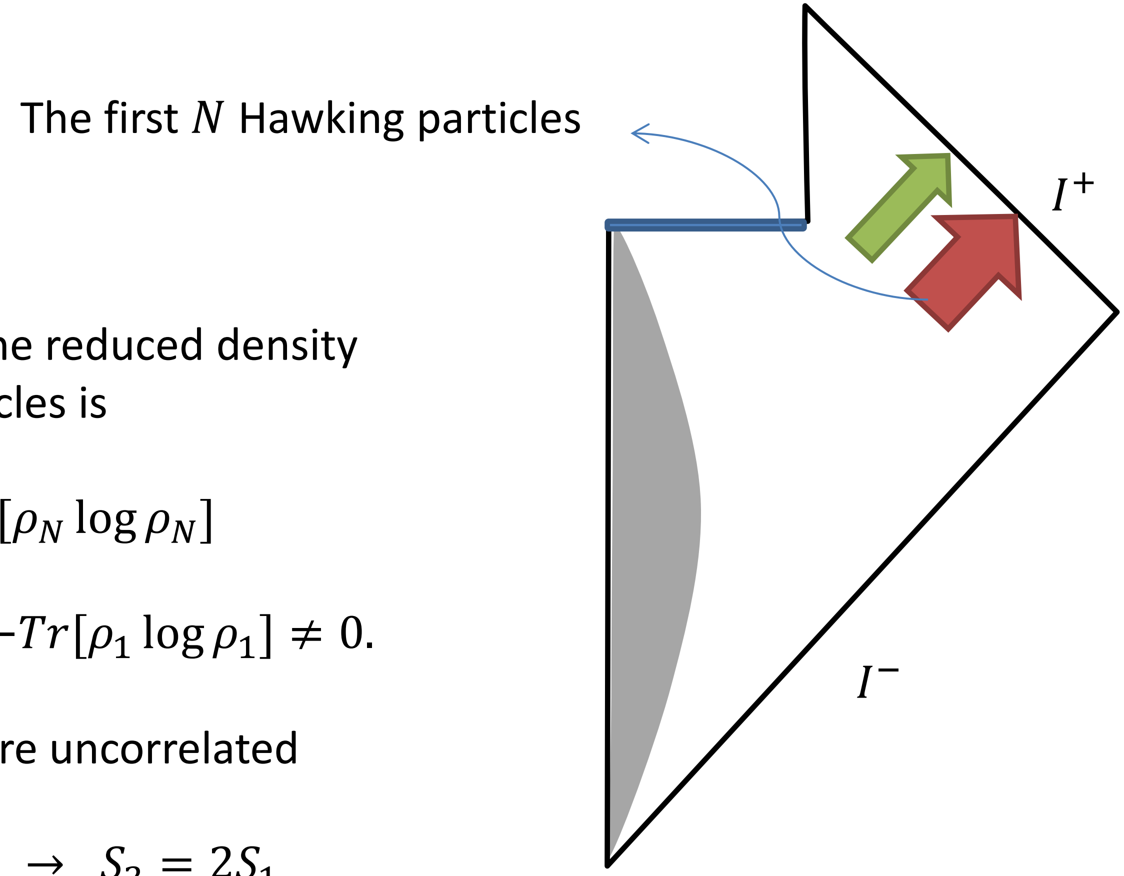
For $N = 2$ because two particles are uncorrelated

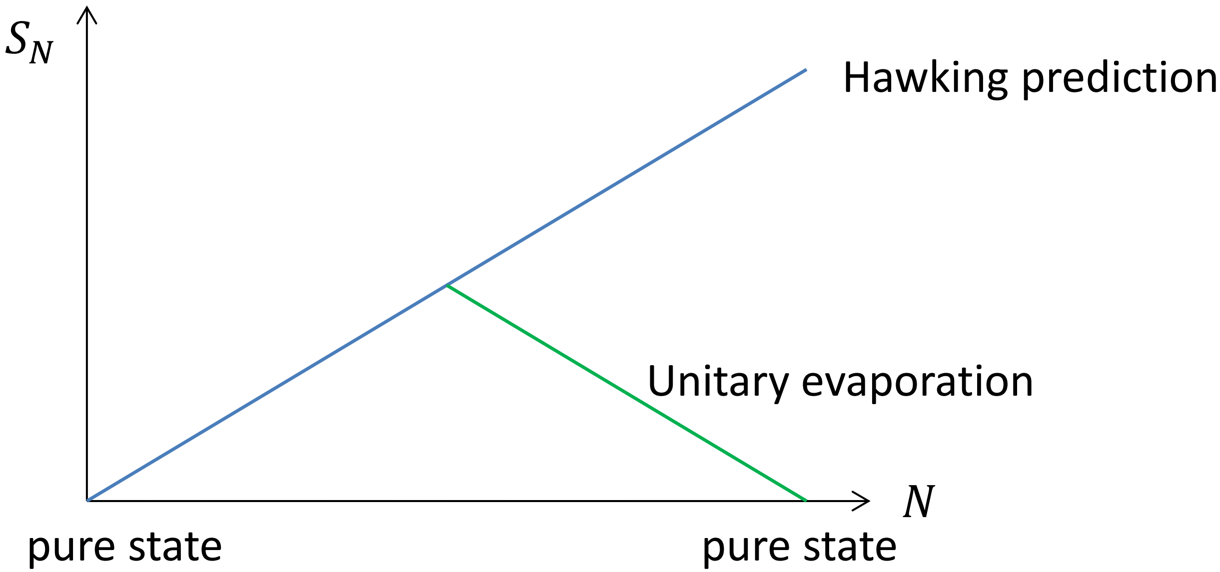
$$\rho_2 = \rho_1 \otimes \rho_1 \;\rightarrow\; S_2 = 2S_1$$

Therefore for N particles

$$\rho_N = \rho_1^N \;\rightarrow\; S_N = NS_1$$

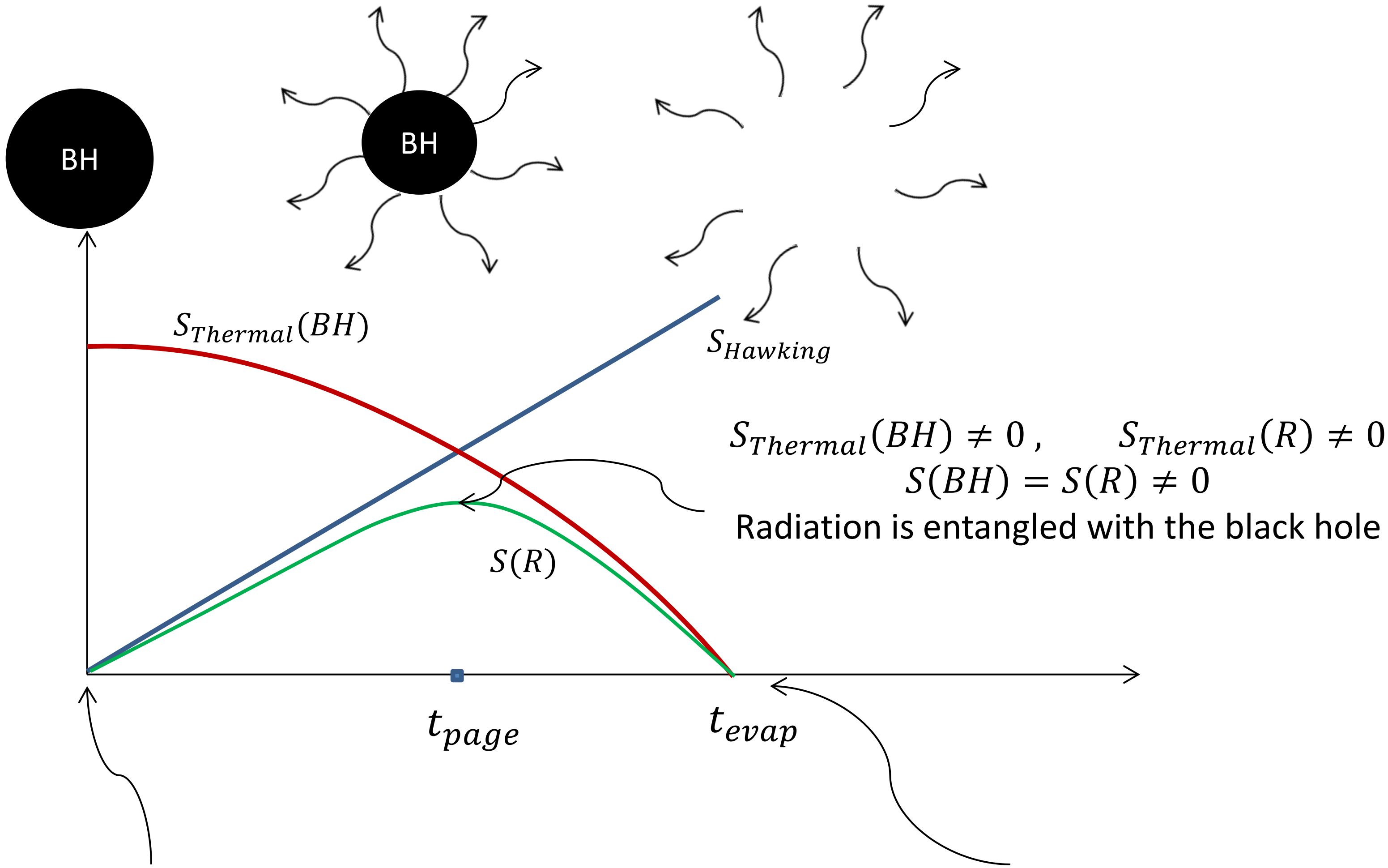
So the entanglement entropy of Hawking radiation increases linearly.





For a unitary evaluation at the late time (after the black hole completely evaporated) the entanglement entropy of radiation has to go to zero. Because we start from a pure state then the Hawking radiation after the evaporation of black hole must be a pure state, i.e. $S_{EE} = 0$.

Page curve



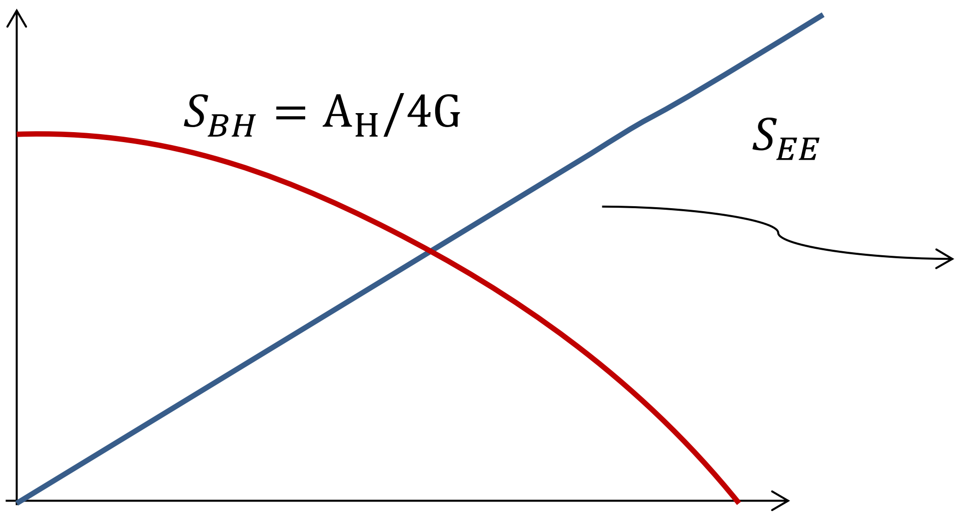
$S_{Thermal}(BH) \neq 0$, $S(BH) = 0$,
$S_{Thermal}(R) = 0$, $S(R) = 0$
$I(BH) = S_{Thermal}(BH)$
Information is stored in microstates of the black hole

$S_{Thermal}(BH) = S(BH) = 0$
$S(R) = 0 \rightarrow$ final pure state, $S_{Thermal}(R) \neq 0$
$I(R) = S_{Thermal}(R)$
Information is stored in correlation functions of the Hawking radiation

Resolving Hawking's paradox

What we learn from Hawking calculations:

- The probability distribution for the number of particles emitted at a particular frequency and angular momentum coincided with a thermal distribution, so the final state is mixed (contradiction with unitary evolution of a quantum system)
- Moreover, particles emitted at one frequency were uncorrelated with those emitted at another frequency.



The size of Hilbert space of the black hole is smaller than the entanglement entropy that one needs to purify the radiation of the black hole.

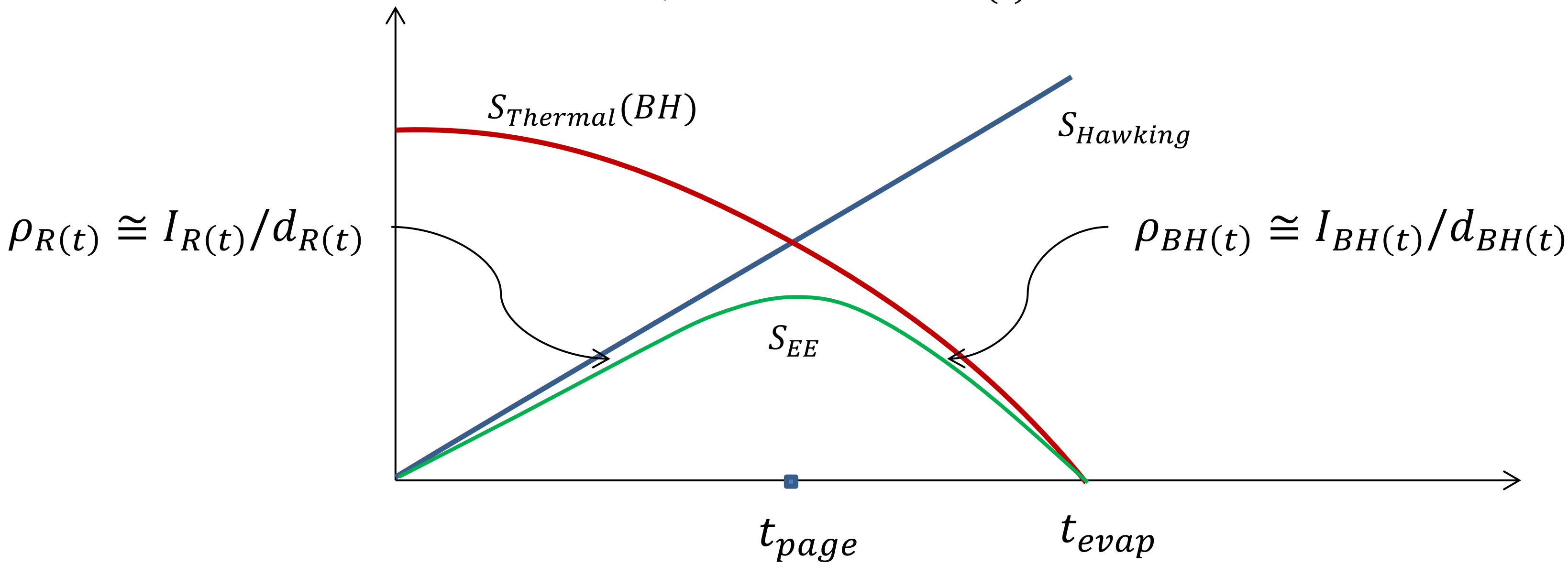
- The root of entanglement entropy of Hawking radiation is the entanglement between black hole as quantum mechanical system and radiation.
- For black hole the size of Hilbert space is determined by area of horizon. As black hole evaporates the size of Hilbert space decreases (area decreases).
- The entanglement entropy of a system is bounded from above by logarithm of dimensionality of a system $S_{EE} < S_{BH}$.

As we stated before, in a random state the density matrix of the smaller subsystem is always maximally mixed. Now consider a random state

$$|\psi\rangle_{\{BH(t),R(t)\}} = U_{random}|\psi_0\rangle$$

and construct the reduced density matrix $\rho_{R(t)}(U)$ then

$$\langle S_{EE}[\rho_{R(t)}(U)]\rangle = \begin{cases} \log d_{R(t)} - \dfrac{d_{R(t)}}{2d_{BH(t)}} + \cdots & d_{R(t)} < d_{BH(t)} \\ \log d_{BH(t)} - \dfrac{d_{BH(t)}}{2d_{R(t)}} + \cdots & d_{BH(t)} < d_{R(t)} \end{cases}$$



Therefore we arrive at the following conclusion.

Hawking was not justified in concluding from thermal distribution that, his results could only be consistent with a mixed state, since mixed and pure states are exponentially close.

The thermal distribution is a leading-order result derived in free field theory. More accurately we should have

$$\langle b^\dagger b \rangle = \frac{1}{e^{\beta\omega} - 1} + O\left(\frac{1}{S}\right) + O\left(e^{-\frac{S}{2}}\right)$$

$O\left(\frac{1}{S}\right)$ = leading order corrections to the Hawking's calculations

$O\left(e^{-\frac{S}{2}}\right)$ = Non-perturbative effects ($S \sim 1/G_N$) needed to make the results perfectly consistent with a pure state.