Introduction to Conformal Field Theory

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Conformal Field Theory

- A quantum Field Theory with conformal invariance; in 2d.
- This invariance is so large that we end up with an integrable quantum field theory; so we need to know about :
  - Quantum Mechanics
  - Quantum Field Theory
  - Symmetry in Field Theory ⇨ Integrability
  - Integrable Quantum Field Theory ⇨ YBE+Quantum groups

Content

Axioms of Quantum Mechanics

1. To every physical state, corresponds a vector (states) in a Hilbert space *H*.

$$\left|\psi\right\rangle \in H$$

2. There exists a unique $\left|0\right\rangle$ vacuum (ground state) , a state in *H* corresponding to “rest” or “nothing”. All physical states have higher energy. And vacuum is normalizable :

$$\left\langle 0 \middle| 0 \right\rangle = 1$$

Axioms of Quantum Mechanics

3. There is the space of linear operators on *H*;

$$D = \{ O : H \to H \}$$

   a. observables are Hermitian operators:

$$O^{+} = O$$

   b. There exists two distinguished operators in D, the Identity: I, and the Hamiltonian: $\hat{H}$

4. The dynamics is given by the Hamiltonian operator

$$i\frac{\partial |\psi\rangle}{\partial t} = \hat{H}|\psi\rangle, \hat{H} \in D$$

Axioms of Quantum Mechanics

5. Symmetries are operators that commute with the Hamiltonian:
$$[S,\hat{H}]=0$$

6. This mean that symmetries annihilate the vacuum:
$$S\left|0\right\rangle=0$$
Otherwise the uniqueness of the vacuum would be challenged,

The Identity operator is also unique.

If the operators S_a are many and they form an algebra the vacuum is annihilated by all the algebra for example the so(3) algebra, and the ground state of the hydrogen atom is an S-wave.

Axioms of Quantum Mechanics

All this goes over to QFT, except for the reduction of state under observation which seems to be problematic:

7. In an experimental observation of the Hermitian operator O, the state is reduced to one of the eignestates of O with probability

$$\left|\left\langle n \middle| \psi \right\rangle\right|^2$$

Quantum Field Theory

Lets move on to axioms of field theory now:

1. Hilbert Space, H
2. Existence and uniqueness of vacuum, Ω in H
3. Underlying manifold (space-time) M, with a metric
4. There exists a unitary representation U, of the group of isometries of M acting on H. Usually this is the Poincare group, but in CFT this is the affine extension of sl(2,R).

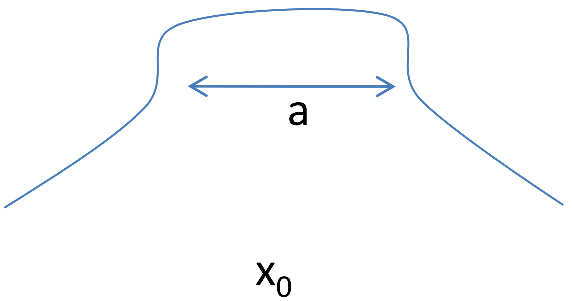
Quantum Field Theory

5. Collection of field operators Φ_a (x), x ε M
   which are distributions . So we need test functions f(x) to use :

$$\Phi(f) = \int dx\, \Phi(x) f(x)$$

A natural question arises as to how much Physics depends on the choice of f, the answer lies in the Renormalization Group equations.
Test functions should remain uniform throughout the manifold; RG governs the change in a:

f(X):



Quantum Field Theory

1. The vacuum should remain invariant under symmetry: $U\Omega = \Omega$
2. The domain of action of the field operators in H is invariant $\Phi(f)H \subset H, \forall \Phi, f$
3. The action of U on Φ is given by conjugation:

$$U(\Lambda)\Phi(x)U(\Lambda)^{+} = \Phi(\Lambda x)$$

In the case of CFT this requirement is relaxed a bit and we allow more complex transformation law.

Quantum Field Theory

4. Locality; if support of *f* and g are space-like then:

$$[\Phi(f),\Phi(g)]=0$$

5. Spectrum: The joint spectrum of the momenta is constrained in the forward light cone.

Quantum Field Theory

Wightman Distributions

Define the n-point correlation functions:

$$W_n = \langle \Omega, \Phi(f_1) \cdots \Phi(f_n), \Omega \rangle$$

Should have various properties, such as invariance under symmetry, positive definiteness, cluster property etc. These are objects of “observation” in experiments.

Euclidean Quantum Field Theory

Wightman Distributions

The n-point correlation functions:

$$W_n = \langle \Omega, \Phi(f_1)\cdots\Phi(f_n), \Omega \rangle$$

Can be analytically continued into the imaginary "time" (changes M from Pseudo-Riemannian into Riemannian)

Conformal Field Theory

Lets us now move on to conformal invariance :

The symmetry of the underlying manifold (space-time) M, is enlarged to so(d,2).

So dilations:

$$x \rightarrow \lambda x$$

and Special conformal transformations are included:

$$x \rightarrow \frac{x - br^2}{1 - 2b.x + r^2 b^2}$$

In d=2, this algebra enlarges to the Virasoro algebra which is the affine extension of sl(2,R), however more care has to be exercised since invariance of correlation functions under an infinite symmetry algebra leaves nothing but a constant behind !

Conformal Field Theory

1. First complexified coordinates are used which mimic the light cone z=t+i x, treated independent of z*, usually referred to as: $\bar{z}$
2. The symmetry of the correlation functions is so(2,2)/Z_2 or sl(2,R)/ Z_2

$$z \rightarrow \frac{az+b}{cz+d}, ab-cd=1$$

3. A similar transformation is required for $\bar{z}$

Conformal Field Theory

4. So the symmetry is really sl(2,R)xsl(2,R).
5. The representation of this symmetry on the Hilbert space and field operators is given by U(g):

$$U(g)\Phi_a(z,\bar{z})U(g)^{+} = \frac{1}{(cz+d)^{2h_a}}\frac{1}{(c'\bar{z}+d')^{2\bar{h}_a}}\Phi_a(gz,g'\bar{z})$$

the conformal weights $h_a, \bar{h}_a$ are generally independent of each other, $h_a - \bar{h}_a = s_a$ is referred to as spin. Should s be integer?

Parafermions

- Parafermions is a name given to fields with fractional Spin
- eg Z_N symmetric theories with central charge $c= 2(N-1)/(N+2)$ (N=2 Ising, N=3 Potts model)
- V. A. Fateev and A. B. Zamolodchikov,Sov. Phys. JETP 62, 215 (1985)
- Parafermions are also related to discrete holomorphicity

Discrete Holomorphicity

- Extend the concept of Cauchy-Riemann conditions to a lattice:

$$\sum_{(ij)\in C} F(z_{ij})(z_j - z_i) = 0$$

- This is the discrete version of the Cauchy integral.
- F is not entirely fixed by these conditions

Discrete Holomorphicity

- Parafermions satisfy these conditions automatically, but only at critical and integrable points of lattice models
- All correlation functions defined in terms of Parafermion fields are discrete holomorphic

Conformal Field Theory

The general holomorphic transformation $z \to w(z)$ acts on the fields as :

$$U(w)\Phi_a(z,\bar{z})U(w)^+ = (\frac{dw}{dz})^{h_a}(\frac{d\overline{w}}{d\bar{z}})^{\bar{h}_a}\Phi_a(w(z),\overline{w}(\bar{z}))$$

However we do not require the correlation functions to be invariant under these transformations, just covariant!

Axiomatic Conformal Field Theory

Now, guided by the axioms of quantum field theory we attempt at constructing a number of axioms for Quantum Conformal Field Theory or CFT for short.

- The basic objects are fields Φ_a defined over the complex plane (M) with signature (+,+). The relevance to equilibrium statistical mechanics makes this signature more popular.
- The fields, or field operators, or operators have a number of properties, they are defined as maps on open subsets of the complex plane.

Axiomatic Conformal Field Theory

- Fields take their values in O(H), (possibly unbounded, mostly self adjoint operators acting on a Hilbert space.
- To be precise these these field operators are defined only on the spaces of test functions on M, ie the Scwartz space $\Im$(M), of rapidly decreasing test functions.
- The matrix coefficients, $\left\langle u, \Phi_a(z)\, v \right\rangle$ where u and v are members of H and <,> is the inner product of H; are well defined.

Axiomatic Conformal Field Theory

- The essential parameters of the theory are the correlation functions, which connect up with experiments and observations:

$$G_{a_1\cdots a_n}(z_1,\cdots,z_n)=\left\langle \Omega,\Phi_{a_1}(z_1)\cdots\Phi_{a_n}(z_n),\Omega\right\rangle$$

- Loosely written as holomorphic functions over the complex plane, ignoring the test functions, and the anti-holomorphic part.

Axiomatic Conformal Field Theory

- They are time ordered products of fields in order to respect causality

$$\mathrm{Re}(z_n) > \mathrm{Re}(z_{n-1}) > \cdots$$

These correlation functions can usually be continued to

$$M_n := \{(z_1, \cdots, z_n) \in C^n, z_i \neq z_j\}$$

Axiomatic Conformal Field Theory

Notation

$$M_n^+ := \{(z_1, \cdots, z_n) \in C^n, z_i \neq z_j, \mathrm{Re}(z_i) > 0\}$$

$$S_n^+ := \{ f \in S(C^n) : Supp(f) \subset M_n^+ \}$$

Here S is a Schwartz space of rapidly decreasing smooth functions

$$S_0^+ := C$$

The group of Mobius transformations Mb

$$z \to \frac{az+b}{cz+d}, ab - cd = 1$$

Osterwalder-Schrader Axioms

Let B be a countable index set, we also use the notation $i=i_1,i_2,\ldots,\ i\ \varepsilon\ B^n$. CFT is describbed by a family of continuous and polynomially bounded correlation functions,

$$G_0 = 1$$

$$G_i : M_n \to C$$

Subject to the following axioms:

Axiom 1 (Locality)

$$\forall\,(i_1,\cdots,i_n)\in B^n,(z_1,\cdots,z_n)\in M_n, \textit{and every permutation } \pi; \textit{we have}$$

$$G_{i_1,\cdots,i_n}(z_1,\cdots,z_n)=G_{\pi(i_1),\cdots,\pi(i_n)}(z_{\pi(1)},\cdots,z_{\pi(n)})$$

Axiom 2 (Invariance)

$$\textit{For } i\in B, \textit{there are conformal weights } h_i,\overline{h_i}\in R, (\textit{not complex cojugates})$$

$$\textit{such that for all } w\in Mb, \textit{and } n\geq 1\ \textit{have}:$$

$$G_{i_1,\cdots,i_n}(z_1,\cdots,z_n;\bar{z}_1,\cdots,\bar{z}_n)=G_{i_1,\cdots,i_n}(w_1,\cdots,w_n;\overline{w}_1,\cdots,\overline{w}_n);$$

$$\textit{with } w_j=w(z_j).$$

Note that if w is a general analytic function then we ask for Covariance and not Invariance

Here $h_a - \overline{h}_a = s_a$ is called the conformal spin and $h_a + \overline{h}_a = d_a$ is called the scaling dimension. We assume that $s_a, d_a \in Z$, see

NS Hawley and M Schiffer "Half Order differentials on Rieman surfaces", Acta. Math. 115(1966)175

The covariance of the fields in axiom 2 corresponds to generalized differential forms under change of coordinates.

By covariance we mean that the correlation function changes under transformation but in a certain specific manner.

$$\Phi(z,\overline{z})(dz)^{h}(d\overline{z})^{\overline{h}}$$

Invariance of Correlation Functions

The invariance axiom severly restricts the form of correlation functions:

1-because of translations all correlation functions depend only on differences $z_i - z_j$

2-Two point functions are determined up to a constant :

$$G_{i,j}(z_1, z_2; \bar{z}_1, \bar{z}_2) = a\delta_{i,j} \frac{1}{(z_1 - z_2)^{h_1}} \frac{1}{(\bar{z}_1 - \bar{z}_2)^{\bar{h}_1}}$$

Examples:

$$h = \bar{h} = 0, \quad G = cons. = 1$$

It is therefore natural to interpret the field with zero conformal weight as identity operator

$$\Phi_0 = I$$

For $h = \bar{h}$ we have:

$$G = a\left|z_1 - z_2\right|^{-4h}$$

LCFT

Note that for the case of $G = -\log|z_1 - z_2|^2$ these axioms hold except for scaling transformation, this is the case of logarithmic conformal field theories where a pair of fields transform into each other and together make the axioms correct.

$$\Phi' = (\tfrac{dw}{dz})^h \Phi(z)$$

$$\Psi' = (\tfrac{dw}{dz})^h \Psi + (\tfrac{dw}{dz})^h \log(\tfrac{dw}{dz}) \Phi$$

This is the case of reducible but not decomposable representations

Gaussian example

The functions G_n are zero if n is odd. They are given by:

$$G_{2n}(z_1,\cdots,z_{2n}) = a \sum_{\sigma\in S_{2n}} \prod_{j=1}^{n} \frac{1}{(z_{\sigma(j)} - z_{\sigma(n+j)})^2}$$

Where S_n is the group of permutations of n objects. Conformal weights are h=1.

Axiom 3 (reflection Positivity)

There is a map *:B -> B, i->i* the * map takes z to its complex conjugate.

1- $G_i(z) = G_{i^*}(-z^*)$

2-for all

$$f \in S^+$$

$$\left\langle f(z_1^*)\cdots, G_{j^* i}(z_1^*, \cdots z_n^*, w_1 \cdots w_n)\, f(w_1)\cdots \right\rangle \geq 0$$

Reconstruction of the Hilbert Space

Axiom 3 provides us with a positive definite hermitian form on the Hilbert space, providing an inner product : $\langle , \rangle$

Reconstruction of the field operators

1- $\forall\ j \in B_0, \quad \Phi_j : S^+ \to End(D)\ is\ linear, \Phi_j\ is\ a\ field\ operator$

$D \subset H\ is\ dense, \Omega \in D, \Phi_j(D) \subset D.$

$Unitary\ representa\,tion\ U\ leaves\ \Omega\ in\mathrm{var}\,iant$

2- Further more the unitary transformation effects the analytic transformation of the operator

$$U(w)\Phi_j(z)U(w)^+ = \left(\tfrac{dw}{dz}\right)^{h_j}\Phi_j(w(z))$$

Reconstruction of the field operators

3-The matrix coefficients can be represented by analytic functions

$$\left\langle \Omega, \Phi_{i_1}(z_{i_1}) \cdots \Phi_{i_n}(z_{i_n}) \Omega \right\rangle = G_i(z)$$

So far what has been said holds for any 2d QFT, what follows is specific to CFT

CFT

- What we set out so far holds for any 2d QFT, so let us now specialise to 2d CFT
- Note that special conformal transformations will not be required ! As it is a consequence of dilatins and conservation of T

CFT

A 2d Quantum Field Theory is a conformal Field Theory if the following holds:

4. The theory is invariant w.r.t. to dilations
5. It has a divergence free energy-momentum tensor
6. It has an associative operator product expansion (OPE) for the primary fields

Dilation Invariance

- The correlation function $G_n(z_1 \cdots z_2)$

Is invariant under scale trans. Z-> λ z, for all arguments.

$$G_n(z_1, \cdots, z_n) = \lambda^{h_1 + \cdots + h_n + \bar{h}_1 + \cdots + \bar{h}_n} G_n(\lambda z_1, \cdots, \lambda z_n)$$

- This implies that the 2-point function has a specific form:

$$G_{i,j}(z_1, z_2; \bar{z}_1, \bar{z}_2) = a\delta_{i,j} \frac{1}{(z_1 - z_2)^{h_1}} \frac{1}{(\bar{z}_1 - \bar{z}_2)^{\bar{h}_1}}$$

- Similar results for 3-pt function and less restrictive on 4-pt function

Conformal Ward Identities

Under the assumption that G is invariant under the Mobius transformations, we have:

$$\sum_j \partial_{z_j} G_i(z_1 \cdots z_n) = 0$$

$$\sum_j (z_j \partial_{z_j} + h_j) G_i(z_1 \cdots z_n) = 0$$

$$\sum_j (z^2{}_j \partial_{z_j} + z_j h_j) G_i(z_1 \cdots z_n) = 0$$

Conformal Ward Identities

The proof of these identities follows from covariance properties of the fields under Mobius transformations. In case of n=4 we observe that these differential equations may be solved to yield:

$$G(z_1, z_2, z_3, z_4) = F(x) \prod_{i<j} z_{ij}^{\frac{1}{3}h - h_i - h_j},$$

$$x = \frac{z_{12} z_{34}}{z_{13} z_{24}}, \qquad h = h_1 + h_2 + h_3 + h_4$$

Anti-holomorphic coordinates follow a similar pattern

Energy-Momentum Tensor

Among the field operators, there must exist 4 operators $T_{\mu\nu}$ which are the generators of the Mobius transformations, they are conserved and traceless. These conditions can be re-written to express it as a single holomorphic operator

$$T_{\mu\nu} = T_{\nu\mu}, \partial_\mu T^{\mu\nu} = 0, T_\mu^\mu = 0$$

$$(\partial_1 + i\partial_2)(T_{11} - T_{22} - 2iT_{12}) = 0$$

$$T(z) = T_{11} - T_{22} - 2iT_{12} = 2(T_{11} - iT_{12})$$

Energy-Momentum Tensor

Clear that the conformal dimension is simple:

$$h(T) = 2, \quad \bar{h}(T) = 0$$

$$h(\bar{T}) = 2, \quad \bar{h}(\bar{T}) = 0$$

$$s(T) = 2$$

Integrable Quantum Field Theory

- Clealry the energy-momentum tensor is a conserved current, $\partial_{\bar{z}} T(z) = 0$
- but there exists an infinity of them

$$\partial_{\bar{z}} T^n(z) = 0, \quad n \in N$$

So we are dealing with an IQFT, albeit a specially simple one !

- You can perturb CFT in an integrable direction, eg mass

Integrable Quantum Field Theory

Perturbing the CFT away from conformal point can still remain integrable, but T will no longer be holomorphic

$$\partial_{\bar{z}} T(z,\bar{z}) + \partial_z U(z,\bar{z}) = 0$$

However an infinite number of conserved currents remain:

$$\partial_{\bar{z}} T^n(z,\bar{z}) + \partial_z U_n(z,\bar{z}) = 0$$

Valid for some n only.

Virasoro Generators

A Laurent expansion of T(z) yields the Virasoro generators:

$$T(z) = \sum_{n \in Z} L_n z^{-2-n}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{c}{12} n(n^2 - 1)$$

Simiral operations for the anti-holomorphic sector hold with a different central charge: $(c, \overline{c})$

So we have the following symmetry $Vir \otimes \overline{V}ir$

Virasoro Generators

The operators L_n and L_n define a unitary representation of $Vir \otimes \overline{V}ir$ over the Hilbert space. In general these representations decompose into irreducible units (Verma Modules) based on highest weight representations, W(c,h)

$$H = \oplus W(c,h) \otimes W(\bar{c},\bar{h})$$

The sum is over a suitable collection of central charge and conformal weights. If there a finite number of terms in the above sum t is called a “minimal” theory.

Primary and Secondary Fields

Field operators should in general have the same covariant properties under the action of the Virasoro algebra as the correlation functions, if these is the case we call the Primary operators:

$$[L_n, \Phi_h(z)] = z^{n+1}\partial_z \Phi_h(z) + h(n+1)\, z^n\ \Phi_h(z)$$

The unit operator would be a simple example with h=0, Secondary fields, which are derived from Primary fields may have a more convoluted transformation, the energy momentum operator is an example.

Operator Product Expansion (OPE)

The divergent nature of QFT and the fact that field operators are distributions leads us to expect that the product of two field operators at the same point becomes divergent. But the product of two field operators is postulated to stay within the algebra of operators. So we have the OPE of two fields:

$$\Phi_{h_1}(z_1)\Phi_{h_2}(z_2) \approx \sum_{k \in B_0} C_{12k}(z_1 - z_2)^{h_k - h_1 - h_2} \Phi_{h_k}\left(\tfrac{z_1+z_2}{2}\right)$$

Operator Product Expansion (OPE)

- The sign $\approx$ means that regular terms have been ignored.
- The scale invariance of the theory has greatly simplified the expansion on the right hand side.
- The structure constants C_{12k} are the central data of any CFT, if given that theory is completely defined.

$$\Phi_{h_1}(z_1)\Phi_{h_2}(z_2) \approx \sum_{k \in B_0} C_{12k}(z_1 - z_2)^{h_k - h_1 - h_2}\Phi_{h_k}(\tfrac{z_1+z_2}{2})$$

Operator Product Expansion (OPE)

The OPE of a Primary field with the energy momentum operator is given by:

$$T(z)\Phi(0) \approx \frac{h}{z^2}\Phi(0) + \frac{1}{z}\frac{\partial}{\partial z}\Phi(z)$$

This is consistent with the action of Ln on Φ. The OPE of T with itself:

$$T(z)T(w) \approx \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{1}{(z-w)}\frac{\partial T}{\partial w}$$

Primary Fields

- The above OPE implies a transformation law for the Primary fields as :

$$\Phi(w) = (\frac{dz}{dw})^h \Phi(z)$$

- This means that Primary fields are meromorphic differential forms:

$$\Phi(z, \bar{z})(dz)^h (d\bar{z})^{\bar{h}} = \Phi(w, \overline{w})(dw)^h (d\overline{w})^{\bar{h}}$$

Verma Module

Assume that the state $|h\rangle$ exists such that :

$$|h\rangle = \lim_{z\to 0} \Phi(z)|\Omega\rangle$$

Easy to show that:

$$L_0|h\rangle = h|h\rangle$$

We can also deduce that: $L_n|h\rangle = 0, n > 0$

We can therefore construct the Verma module:

$$\{L_{-n_1} \cdots L_{-n_k}|h\rangle = 0, n_j > 0, k \in N\}$$

Conformal Family

The descendent states can $L_{-n_1} \cdots L_{-n_k} \left| h \right\rangle$ be viewed as excited states.

Correspondingly a conformal family $[\Phi_h]$ can be constructed by: $L_{-n_1} \cdots L_{-n_k} \Phi_h$

State-Operator correspondence:

To each state $\left| v \right\rangle$ there corresponds an operator Φ_v (need not be primary) such that

$$\left| v \right\rangle \in H, \quad \Phi_v \left| 0 \right\rangle = \left| v \right\rangle$$

Operator Product Expansion (OPE)

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$$\Phi_{h_1}(z_1)\Phi_{h_2}(z_2) \approx \sum_{k \in B_0} C_{12k}(z_1 - z_2)^{h_k - h_1 - h_2} \Phi_{h_k}\left(\tfrac{z_1+z_2}{2}\right)$$

The Central Data of a CFT

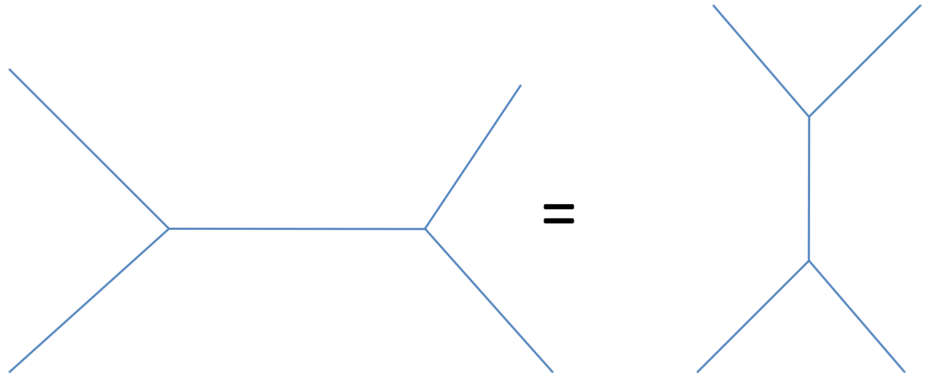
- A CFT is completely determined if we know the following:
  - The set of primary fields and their conformal dimensions for a given central charge
  - The set of structure coefficients for the OPE of the primary fields.

Bootstrap Idea

If the OPE is required to be associative strong conditions on the structural coefficients are imposed which can be used to solve for them:

$$(\Phi_a \Phi_b)\Phi_c = \Phi_a(\Phi_b \Phi_c) \Rightarrow \sum_d C_{ab}{}^d C_{dc}{}^e = \sum_d C_{bc}{}^d C_{da}{}^e$$

Pictorially this relationship can be presented as a crossing invariance in scattering amplitude:



Bootstrap Idea

The 4-pt function can be exactly calculated using the bootstrap idea, since the structure coefficient can be computed (at least in a minimal model).

Use the Mobius transformation to shift the four pints to z_1 =z,z_2 =0, z_3 =infnity,z_1=z, then the four point function is a function z and $\bar{z}$.

$$G_{ijlm}(z,\bar{z}) = \sum_k C_{ij}^k C_{lm}^k F_k(z)\bar{F}_k(\bar{z})$$

Bootstrap Idea

The functions F_k are called Conformal Blocks

In case of minimal models there are a finite number of conformal blocks.

Fusion Rules

The OPE may be written in a compact notation:

$$[\Phi_a]\times[\Phi_b] = N_{ab}^{c}[\Phi_c]$$

Where bu the notation $[\Phi_a]$ we mean the entire descendents of the primary field $[\Phi_a]$, a family. The positive integers N_{ab}^{c} indicate the number of occurrences of elelments of the family c in OPE of family a by family b.

Vertex Operator Algebra (VOA)

The mathematical concept of a VOA was introduced by Borcherds (1986) which has proved to be extremely useful. The associatively of the OPE is automatically encoded in associatively of VOA. Here we shall offer an intuitive treatment of the subject.

VOA

A vertex operator algebra is a

- vector space H, with a distinguished vector (the vacuum). $|\Omega\rangle \in H$
- An endomorphism $U(z) = e^{zL_{-1}}$, the translation operator. $U \in End(H)$

.

VOA

- A linear map from H to the space of field operators F (the state-operator correspondence)

$$\Phi(z) = \sum_{n \in Z} \Phi_{(n)} z^{-n-1}, \Phi_{(n)} \in End(H)$$

$$\Phi : H \rightarrow F$$

Essentially means that a field operator is given by its Laurent expansion coefficients

Such that:

VOA

1. Translation Covariance: (move the fields on the complex plane)

$$U(w)\Phi(z)U(-w) = \Phi(z+w)$$

2. Locality: (commutators of field is well behaved) for large enough N

$$(z-w)^N[\Phi_a(z), \Phi_b(w)] = 0$$

VOA

Vacuum is

- invariant: $U(z)\Omega = \Omega$

- Corresponds to the identity on H $I_H \Rightarrow \Omega$
- Generates the state-operator correspondence

$$\Phi_a(z)\left|\Omega\right\rangle \xrightarrow{z\to 0} \left|\Phi_a\right\rangle$$

Heisenberg VOA

- This is the VOA of the free boson, consider the Heisenberg algebra: (C is central)

$$[a_n, a_m] = m\delta_{n+m}C$$

The field operator is given by:

$$\Phi(z) = \sum_n a_n z^{-n-1}$$

Locality:

$$\Phi(z)\Phi(w) \approx \frac{C}{(z-w)^2}$$

Heisenberg VOA

- translation $U(w)\Phi(z)U^{-1}(w) = \Phi(z+w)$

$$\Phi(z+w) = \sum_n a_n (z+w)^{-n-1}$$

$$U(z) = Exp(z\sum_m a_{-m-1}a_m)$$

$$L_n = \sum_m a_{-m+n}a_m$$

Conformal VOA

- There exists a generating field (the energy-momentum tensor)

$$L(z) = \sum_n L_n z^{-n-2}$$

- The energy-momentum is given by

$$L(z)\Omega \xrightarrow{z \to 0} L_{-2}\Omega$$

Boundary CFT (BCFT)

- BCFT is defined over a region of the complex plain with boundary conditions imposed on the field operators
- In particular the energy momentum tensor has to be defined such that we don't get energy flow out of the region: $T_{\mu\nu}n^{\nu}=0$
- Here n^{ν} is the vector orthogonal to the boundary

BCFT

- This bc has the effect that it changes the Laurent expansion of T : relating the L_n to $\bar{L}_n$
- This means that the symmetry is reduced:

$$Vir \otimes \overline{V}ir \Rightarrow Vir$$

- We can always map the compact region on the complex plane to the upper half plane and impose bc on the real line :

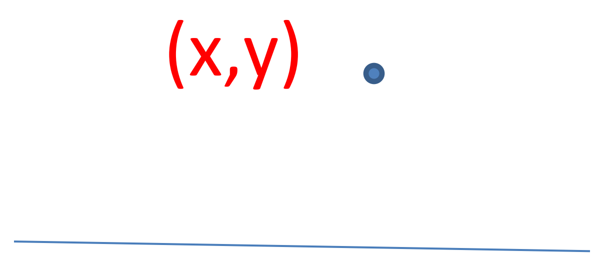
$$T(z) = T(z^*) = 0 \quad \Rightarrow \quad L_n = \bar{L}_{-n}$$

BCFT

This has the offshoot that correlation functions will combine holomorphic and anti-holomorphic parts, in particular the one point functions need no longer vanish near a boundary:

$$\left\langle \Phi(z,\bar{z}) \right\rangle \approx y^h$$

the conformal weights are related to critical exponents near surfaces



Conformal Quantum Mechanics: CQM

1. We have a finite number of degrees of freedom, hence a finite algebra is enough to make it integrable.
2. To mimic conformal invariance need scale invariance, so the minimal set of transformations are :

$$t \to t + a : \quad H = \frac{\partial}{\partial t}$$

$$t \to \lambda\, t \quad : \quad D = t\,\frac{\partial}{\partial t}$$

$$[D, H] = -H$$

Conformal Quantum Mechanics

3. We cannot get far without inversions:

$$K = t^2 \frac{\partial}{\partial t}$$

Together they form the Mobius transformations in time; so(2,1):

$$[D,H] = -H, [K,H] = -2D, [D,K] = K$$

$$t \to \frac{at+b}{ct+d}, ab - cd = 1$$

Conformal Quantum Mechanics

q(t) is a conformal primary operator with dimension *h*, if:

$$q(\lambda\, t) = \lambda^h\, q(t)$$

Or

$$[D, q] = t\,\tfrac{d}{dt}\, q + h\, q$$

$$\langle q_h(t) q_{h'}(t') \rangle \approx (t - t')^{-2h}\, \delta_{h,h'}$$

Conformal Quantum Mechanics

- There is clearly a problem with normalizability since

$$\langle q^2 \rangle \to \infty$$

- All this has been done in Heisenberg Picture and we need to extend to Schrodinger picture what is the Hamiltonian H which implements time translation, preferably three conserved vectors.

Conformal Quantum Mechanics

- Require the symmetry group to be so(2,1):

$$[D,H]=-iH,[D,K]=iK,[K,H]=-2iD.$$

Where H is the Hamiltonian, D is the dilations, and K is the special conformal transformation. Clearly the symmetry operators do not commute with H, nevertheless we require the vacuum to be unique so it must be annihilated by all three operators. Here we face a problem.

This problem does not exist in the Hydrogen atom because the Hamiltonian is the Casimir and commutes with all symmetry operators

CQM

- The vacuum is annihilated by all three operators
- The primary staes are taken to be eigenvectors of D

$$D|h\rangle = h|h\rangle$$

$$H|h\rangle = |h-1\rangle, \quad K|h\rangle = |h+1\rangle$$

- If a highest weight h-1 cannot exist.
- For the Vacuum h=0 and highest weight, so forces K also to vanish.

Conformal Quantum Mechanics

Consider the Hamiltonian:

$$H = \frac{1}{2}(p^2 + \frac{g}{q^2}), \quad g > 0, \quad [p,q] = -i$$

The wave functions are scale invariant:

$$\langle q | \psi \rangle = \psi(q,t) = \psi(\lambda q, \lambda^2 t)$$

The generator of dilations is:

$$D = tH - \frac{1}{4}(pq + qp)$$

The special conformal transformation is:

$$K = -t^2 H + 2t\, D + \frac{1}{2} q^2$$

THANK YOU

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