Effective Potential in Tortoise Coordinate

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Contents

1 Tortoise Coordinate

- Suppose that the equation of motion of a general field $\Phi(t, r; k_i) = e^{-i\omega t}\phi(r; k_i)$ in a spherically symmetric space has been written as

$$\frac{d^2}{dr^2}\phi(r) + A(m, \omega, k_i; r)\frac{d}{dr}\phi(r) + \left(F^2(r)\omega^2 + B(m, \omega, k_i; r)\right)\phi(r) = 0 \quad (1)$$

  k_i are some quantum numbers, $A(r), B(r)$ and $F(r)$ are some general functions appeared in equations of motion.

- T(ortoise) coordinate is defined by new variable r^* and new field ϕ^* as

$$\phi^*(r^*) = \theta(r)\phi(r), \qquad r^* = f(r), \tag{2}$$

  where the functions $f(r)$ and $\theta(r)$ can be uniquely determined such that one be able to bring the equation of motion of ϕ field in Schrodinger-like form

$$(-\frac{d^2}{dr^{*2}} + U(r^*))\phi^*(r^*) = \omega^2\phi^*(r^*) \tag{3}$$

- We see that a Tortoise observer find the dynamics of a field is non-relativistically.

2 Why This Frame Is Important?

Understanding the physics in Tortoise frame is important for several reasons.

1. The equation of motion in this coordinate is in Schrodinger-like form which is familiar. One may be able to solve it exactly and find the spectrum of the theory and construct the Hilbert space of the theory.
2. Imposing the appropriate boundary condition on various fields needs knowing of the asymptotic behavior of that fields. Finding potential U in Totoise frame sheds light on this problem.
3. ...

3 General Form of Effective Potential

- Applying Tortoise change of variables and demanding $F^2\omega^2 \hookrightarrow \omega^2$ and noting $\frac{d}{dr^*}\phi(r^*) \hookrightarrow 0$, we obtain two following conditions

$$
\begin{aligned}
f' &= \pm F, && (4)\\
\frac{f''}{f'} &= 2\frac{\theta'}{\theta} - A. && (5)
\end{aligned}
$$

- Then, the general form of effective potential in spherically symmetric space reads as

$$
U(r) = \frac{1}{2F^2}\left(\frac{F''}{F} - \frac{3}{2}(\frac{F'}{F})^2 + A' + \frac{A^2}{2} - 2B\right) \tag{6}
$$

- Note that potential in which was written as (6) is a function of r instead of r^*. So we have to solve $f' = \pm F$ and then find the inverse function $r = f^{-1}(r^*)$.

- This procedure, however, can not be done analytically for all cases.

- Our aim is analyzing the global behavior of the effective potential, by some simple calculations such as finding the extremums of potential, and asymptotic behavior of U without using the exact coordinate relation $r = f^{-1}(r^*)$.

- Note also that there is a freedom in choosing the sign of $f'(r)$. The form of the potential does not depend on this sign but the Continuity of potential functions inside and outside of the horizon usually needs to choose different signs for these two regions.

4 Scalar in 3-D Black Hole Background

Now, we drive the E.o.M of scalar fields in a 3D spherically symmetric black hole background. Then we obtain explicit form of the potential in Tortoise coordinate.

- Consider the following spherically symmetric form for the metric in three dimension

$$ds^2 = -N^2(r)dt^2 + R^2(r)\left(d\theta + N_\theta(r)dt\right)^2 + P^2(r)dr^2, \tag{7}$$

  The $N(r), R(r), N_\theta(r)$ and $P(r)$ are specified with the specific solution of E.o.M of gravity.

- The equation of motion of a massive scalar field in curved space-time is given by

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial_\mu - m^2)\right)\Phi(t,\theta,r) = 0, \tag{8}$$

- Using (7), (8) and setting $\Phi(t,\theta,r) = e^{-i(\omega t - k\theta)}\phi(r)$ then

$$\frac{\partial^2\phi}{\partial r^2} + \left(\frac{\Delta'}{\Delta} - 2\frac{P'}{P}\right)\frac{\partial\phi}{\partial r} + \left(\frac{P^2}{N^2}(\omega + kN_\theta)^2 + \frac{P^2}{R^2}(m^2R^2 - k^2)\right)\phi = 0 \tag{9}$$

  where $\Delta = \sqrt{-g} = NRP$

- The form of the function $U(r)$ for scalar field using A and B can be written as

$$U(r) = \frac{N^2}{2P^2}\left(\frac{R''}{R} - \frac{R'}{R}\left(\frac{1}{2}\frac{R'}{R} + \frac{P'}{P} - \frac{N'}{N}\right) - 2B\right) \tag{10}$$

In the next part of this section, we will present some examples, BTZ black hole, new type black hole and black hole with no horizon.

We also evaluate the potential for zero mode, $k = 0$.

4.1 BTZ Black Hole

The BTZ black hole solution is given by

$$\begin{aligned} N^2(r) &= \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}, & R^2(r) &= r^2 \\ P^2(r) &= \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}, & N_\theta &= \frac{r_+ r_-}{r^2} \end{aligned} \tag{11}$$

where r_+ and r_- are the horizons of the BTZ black hole.

a) *Extremal BTZ:*

- First of all, we calculate $r^* = f(r)$ as

$$r^* = \pm \int \frac{P}{N} dr = \pm \int \frac{r^2}{(r^2 - r_0^2)^2} dr \tag{12}$$

- By appropriate choosing of the sign in front of P/N, $-$ for $r > r_0$ and $+$ for $r < r_0$ regions, and setting the integration constant to zero we have

$$\begin{array}{lcl} r^* = \pm \dfrac{r}{2(r_0^2 - r^2)} \pm \dfrac{1}{4r_0} \ln{(\pm \dfrac{r_0 - r}{r_0 + r})} & & \\ r = [0, r_0) & \Leftrightarrow & r^* = [0, +\infty), \\ r = (r_0, +\infty) & \Leftrightarrow & r^* = (+\infty, 0]. \end{array} \tag{13}$$

where $+$ and $-$ are understood for interior and exterior regions of black hole respectively.

- For potential we obtain

$$U(r) = \frac{(r^2 - r_0^2)^2}{4r^6} \left((3 - 4m^2) r^4 + 2r_0^2 r^2 - 5r_0^4 \right). \tag{14}$$

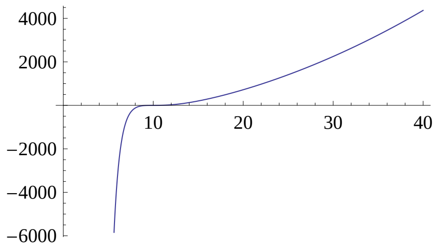

Figure 1: Potential for massless scalar in extremal BTZ: $r_0 = 1$

- The extremums of $U(r^*)$

$$\frac{d}{dr^*}U(r^*) = \frac{(r^2 - r_0^2)^2}{2r^9}\left((3-4m^2)r^8 + (6+4m^2)r_0^4 r^4 - 24r_0^6 r^2 + 15r_0^8\right) = 0 \quad (15)$$

The asymptotic behavior of $U(r)$ and the extremums of $U(r)$ depend on the range of m^2.

1. $m^2 = 0$: In this case

$$U(r \mapsto 0) \mapsto -\infty, \quad U(r = r_0) = 0, \quad U(r \mapsto +\infty) \mapsto +\infty. \quad (16)$$

From (15) we see $U(r)$ has only one real positive extremum in $r = r_0$. In fact, the $r = r_0$ also is an inflection point of $U(r)$(Figure(1)).

2. $0 \leq m^2 \leq \frac{3}{4}$:

$$U(r \mapsto +\infty) \mapsto +\infty. \quad (17)$$

and potential has two extremums one in $r_1 = r_0$ and the other in $r_2 > r_0$(Figure(2)).

3. $\frac{3}{4} < m^2 \leq \frac{3.33}{4}$:

$$U(r \mapsto +\infty) \mapsto -\infty. \quad (18)$$

and potential has three extremums one in $r_1 = r_0$ and two others are in $r_2, r_3 > r_0$(Figure(1)).

4. $\frac{3.33}{4} < m^2$:

$$U(r \mapsto +\infty) \mapsto -\infty. \quad (19)$$

and potential has only one extremun in $r_1 = r_0$(Figure(4)).

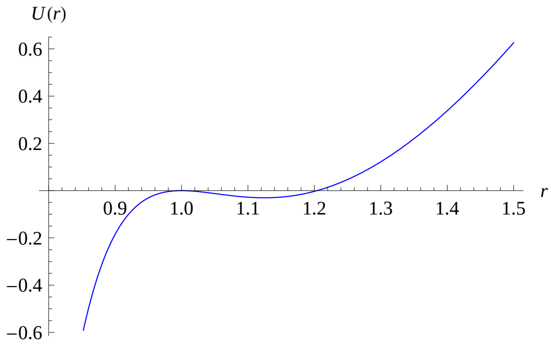

Figure 2: Potential for massive scalar $0 \leq m^2 = 0.5 \leq \frac{3}{4}$ in extremal BTZ: $r_0 = 1$

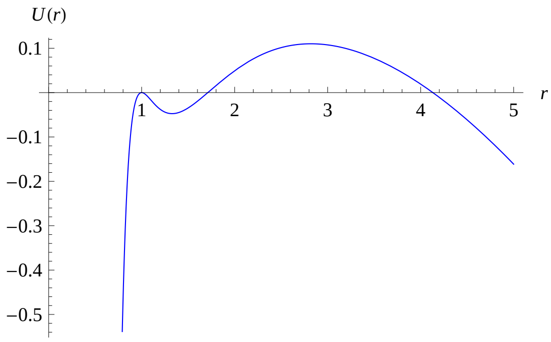

Figure 3: Potential for massive scalar $\frac{3}{4} < m^2 = \frac{3.1}{4} \leq \frac{3.33}{4}$ in extremal BTZ: $r_0 = 1$

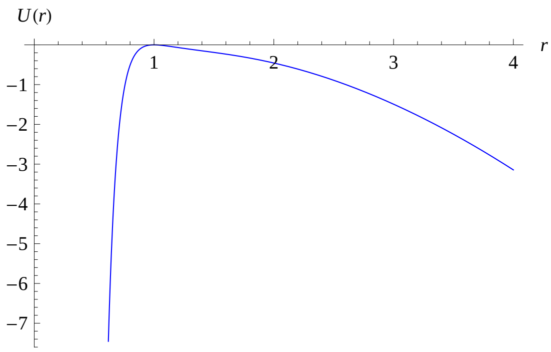

Figure 4: Potential for massive scalar $\frac{3.33}{4} < m^2 = 1$ in extremal BTZ: $r_0 = 1$

b)*non-Extremal BTZ:*

- In this case, after finding r^* in term of r, by appropriate choose of the sign of $+$ and $-$, we have

$$\begin{array}{lcl} r = [0, r_-) & \Leftrightarrow & r^* = [0, +\infty), \\ r = (r_-, r_+) & \Leftrightarrow & r^* = (+\infty, -\infty), \\ r = (r_+, \infty) & \Leftrightarrow & r^* = (-\infty, 0], \end{array} \tag{20}$$

- The effective potential also reads as

$$U(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{4r^6} \left((3 - 4m^2)r^4 + (r_+^2 + r_-^2)r^2 - 5r_+^2 r_-^2\right). \tag{21}$$

  Let us consider only the massless scalar.

- $U(r)$ has three real positive roots at $r_\pm$ and

$$r_0 = -\frac{1}{6}(r_+^2 + r_-^2 - \sqrt{r_+^4 + r_-^4 + 62 r_+^2 r_-^2})$$

  where $r_- < r_0 < r_+$.

- The asymptotic behavior of U is such that $U \to -\infty$ as $r \to 0$ and $U \to +\infty$ as $r \to +\infty$.

- The extremums of $U(r^*)$ is given by solving the following equation

$$\frac{d}{dr^*}U(r^*) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{2r^9} \times$$
$$\times \left(3r^8 + (r_+^4 + r_-^4 + 4r_+^2 r_-^2)r^4 - 12 r_+^2 r_-^2 (r_+^2 + r_-^2) r^2 + 15 r_+^4 r_-^4\right) = 0. \tag{22}$$

  Again r_+ and r_- are extremums of $U(r^*)$.

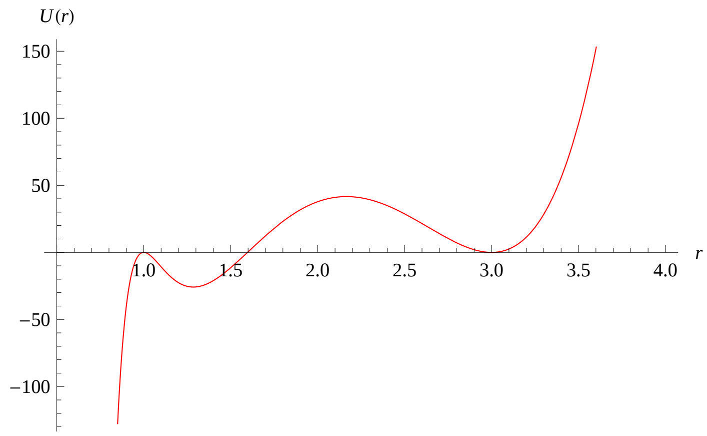

Figure 5: Potential for massless scalar in non-extremal BTZ: $r_- = 1, r_+ = 3$

- Defining $x = r^2$ and recalling properties of quartic equation, we can prove that there are exactly two real positive extremas for potential.

- In fact, if x_1, x_2, x_3 and x_4 be the roots of (22) they should satisfy

$$\begin{aligned} & x_1 x_2 x_3 x_4 = 15 r_+^4 r_-^4 > 0 \\ & \Sigma x_i x_j x_k = 12 r_+^2 r_-^2 (r_+^2 + r_-^2) > 0 \\ & \Sigma x_i x_j = r_+^4 + r_-^4 + 4 r_+^2 r_-^2 > 0 \\ & \Sigma x_i = 0 \end{aligned} \tag{23}$$

Note that if a complex number z be a solution of a polynomial equation then the $\bar{z}$ also is a solution.

We may plot the $U(r)$ qualitatively as(Figure(5))

4.2 Black Hole with No Horizon

In this section, we study black holes with vanishing horizon in three dimensions.

As we will see, the global behavior of this solution, in general, is different from the BTZ and new type solutions. Especially, the asymptotic behavior of potential is completely different.

- A class of such solutions in three dimensional massive gravity is given by

$$\begin{aligned} R^2(r) &= l^2 a_+ r^2 + 2r + l^2 d, \qquad N^2 = \frac{4r^2}{l^2 R^2} \\ P^2(r) &= \frac{l^2}{4r^2} \qquad N_\theta = \frac{2r}{lR^2} \end{aligned} \tag{24}$$

  where l, m are parameters in three dimensional massive gravity and a_+, d are some integration constants.

- Having regular solution forces us $m^2 l^2 = 17/2$ and $a_+ > 0, d > 0$.

- Defining $l^2 a_+ r = x$ and $l^4 d a_+ = s$ we obtain $U(r)$ for *massless* scalar as

$$U(r) = \frac{4x^3}{a_+ l^6 (x^2 + 2x + s)^3} \big(x^3 + 6x^2 + (6s+3)x + 4s\big) \tag{25}$$

- We see that $U(r) \geq 0$ for $r \geq 0$ and

$$U(r \to 0) \to 0, \qquad U(r \to \infty) \to \frac{4}{a_+ l^6} \tag{26}$$

- For derivative of potential $\frac{d}{dr^*}U(r^*)$

$$\frac{d}{dr^*}U(r^*) \propto \frac{x^4}{R^9}\left((3-s)x^3 + (5s+1)x^2 + 2s(2s+1)x + 2s^2\right) = 0. \tag{27}$$

- We see that $x = 0$ and roots of

$$(3-s)x^3 + (5s+1)x^2 + 2s(2s+1)x + 2s^2 = 0 \tag{28}$$

  are the extremums of U.

- Here there are two regions in term of s.

  1. If $0 < s \leqq 3$, equation (28) has no solution.
  2. If $s > 3$, then by considering the general properties of cubic equations

$$\begin{aligned} x_1x_2x_3 &= -\frac{2s^2}{3-s} > 0 \\ \Sigma x_i x_j &= \frac{2s(2s+1)}{3-s} < 0 \\ \Sigma x_i &= -\frac{5s+1}{3-s} > 0 \end{aligned} \tag{29}$$

     one can prove that (28) has only one real positive solution.

  So, the general behavior of the effective potential can be plotted as (Figure(9)).

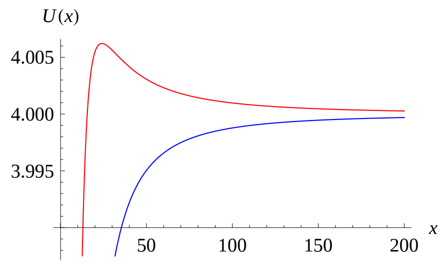

Figure 6: Potential for massless scalar in black hole with no horizon: the blue curve is for $0 < s \leq 3$ and the red curve is for $s > 3$.

5 Fermions in $3-$D Black Hole Background

- The equation of motion for a spinor field is Dirac equation in curved background

$$(-i\gamma^a e_a^\mu D_\mu + m)\Psi(t,\theta,r) = 0, \tag{30}$$

where the covariant derivative is defined as $D_\mu = \partial_\mu - \frac{i}{4}\eta_{ac}\omega^c_{b\mu}\sigma^{ab}$ and $\sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$.

$\omega^c_{b\mu}$ are spin connection and are defined by veilbeins and Christoffel coefficients as $\omega^c_{b\mu} = e^c_\nu \partial_\mu e^\nu_b + e^c_\nu e^\sigma_b \Gamma^\nu_{\sigma\mu}$.

- For the background (7), one can obtain non-zero veilbains as

$$e^t_0 = -\frac{1}{N}, \qquad e^\theta_0 = \frac{N_\theta}{N}, \qquad e^r_1 = \frac{1}{P}, \qquad e^\theta_2 = \frac{1}{R}, \tag{31}$$

and so the covariant derivatives take such forms

$$\begin{aligned} D_t &= \partial_t, \qquad D_\theta = \partial_\theta, \\ D_r &= \partial_r - \frac{i}{8NR}\partial_r(R^2 N_\theta)\sigma_1 \end{aligned} \tag{32}$$

- By rewriting two dimensional spinor field $\Psi(t,\theta,r)$ as

$$\Psi(t,\theta,r) = e^{-i(\omega t - k\theta)}\begin{pmatrix}\phi^+ \\ \phi^-\end{pmatrix} \tag{33}$$

we find two coupled equations for ϕ^+ and ϕ^-

$$\begin{aligned} \mathcal{M}\phi^+ + \mathcal{N}\phi^- + \partial_r\phi^- = 0 \\ \bar{\mathcal{M}}\phi^- + \bar{\mathcal{N}}\phi^+ + \partial_r\phi^+ = 0 \end{aligned} \tag{34}$$

where

$$\begin{aligned} \mathcal{N} &= i\frac{P}{N}(\omega + kN_\theta) \\ \mathcal{M} &= \frac{1}{8NR}\partial_r(R^2N_\theta) + P(m + i\frac{k}{R}), \end{aligned} \tag{35}$$

- One may obtain following tow decoupled equations for ϕ^+ and ϕ^-

$$\partial_r^2\phi^+ + (-\frac{\partial_r\bar{\mathcal{M}}}{\bar{\mathcal{M}}})\partial_r\phi^+ + (|\mathcal{N}|^2 - |\mathcal{M}|^2 + \bar{\mathcal{M}}\partial_r(\frac{\bar{\mathcal{N}}}{\bar{\mathcal{M}}}))\phi^+ = 0, \tag{36}$$

$$\partial_r^2\phi^- + (-\frac{\partial_r\mathcal{M}}{\mathcal{M}})\partial_r\phi^- + (|\mathcal{N}|^2 - |\mathcal{M}|^2 + \mathcal{M}\partial_r(\frac{\mathcal{N}}{\mathcal{M}}))\phi^- = 0 \tag{37}$$

- Again the coefficient of ω^2 is $(\frac{P}{N})^2$. This is just from the fact that spinor equation is obtained in a spherically symmetric space.

- Thus, we obtain

$$U(r) = \frac{N^2}{2P^2}\left(\frac{P''}{P} - \frac{N''}{N} - \frac{\bar{\mathcal{M}}''}{\bar{\mathcal{M}}} - \frac{3}{2}(\frac{P'}{P})^2 + \frac{1}{2}(\frac{N'}{N})^2 + \frac{3}{2}(\frac{\bar{\mathcal{M}}'}{\bar{\mathcal{M}}})^2 + (\frac{P'}{P})(\frac{N'}{N}) - 2B\right) \tag{21}$$

for ϕ^+ and $\bar{U}$ for ϕ^-.

Thanks A Lot For Your Attention.