D-brane couplings at order $O(\alpha'^2)$ from T-duality and S-duality

Based on

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1 Introduction

The dynamics of D-branes is well-approximated by the effective world-volume field theories which consist of the sum of Chern-Simons (CS) and Dirac-Born-Infeld (DBI) actions,*i.e.*,

$$S = S_{DBI} + S_{CS}$$

where at order $O(\alpha'^0)$ they are

$$\begin{aligned} S_{DBI} &\supset -T_p \int d^{p+1}x\, e^{-\phi}\sqrt{-\det\left(G_{ab}+B_{ab}+2\pi\alpha' F_{ab}\right)} \\ S_{CS} &\supset T_p \int_{M^{p+1}} C \wedge (e^{2\pi\alpha' F+B}) \end{aligned}$$

The curvature corrections to CS part have been found by requiring that the chiral anomaly on the world volume of intersecting D-branes (I-brane) cancels with the anomalous variation of the CS action, *i.e.*,

$$S_{CS} \supset T_p \int_{M^{p+1}} C \wedge (e^{2\pi\alpha' F+B}) \wedge \left(\frac{\mathcal{A}(4\pi^2\alpha' R_T)}{\mathcal{A}(4\pi^2\alpha' R_N)}\right)^{1/2} \tag{1}$$

where M^{p+1} represents the world volume of the D$_p$-brane, C is meant to represent a sum over all appropriate RR potential and $\mathcal{A}(R_{T,N})$ is the Dirac roof genus of the tangent and normal bundle curvatures respectively,

$$\sqrt{\frac{\mathcal{A}(4\pi^2\alpha' R_T)}{\mathcal{A}(4\pi^2\alpha' R_N)}} = 1 + \frac{(4\pi^2\alpha')^2}{384\pi^2}(\mathrm{tr}R_T^2 - \mathrm{tr}R_N^2) + \cdots \tag{2}$$

For totally-geodesic embeddings of world-volume in the ambient spacetime, $\mathrm{R}_{T,N}$ are the pulled back curvature 2-forms of the tangent and normal bundles respectively.

The curvature corrections at order $O(\alpha'^2)$ to the DBI action have been found by requiring consistency of the effective action with the $O(\alpha'^2)$ terms of the corresponding disk-level scattering amplitude. For totally-geodesic embeddings of world-volume in the ambient spacetime, the corrections in string frame for zero $B_{\mu\nu}, F_{ab}$ and for constant dilaton are [1]

$$S_{DBI} \supset \int d^{p+1}x\, e^{-\phi}\sqrt{-G}\left(R_{abcd}R^{abcd} - 2\hat{R}_{ab}\hat{R}^{ab} - R_{abij}R^{abij} + 2\hat{R}_{ij}\hat{R}^{ij}\right)$$

where $\hat{R}_{ab} = G^{cd}R_{cadb}$ and $\hat{R}_{ij} = G^{cd}R_{cidj}$. Here a tensor with the world-volume or transverse space indices is the pulled back of the corresponding bulk tensor onto world-volume or transverse space. For the case of D_3-brane with trivial normal bundle the above curvature couplings have been modified to include the complete sum of D-instanton corrections by requirement the $SL(2,Z)$ invariance of the couplings.

In above actions the B-field is assumed to be constant. We are going to add new couplings involving the non-constant B-field by requiring the couplings to be consistent with T-duality. Moreover we add more couplings to makes them consistent with S-duality.

2 T-duality

The full set of nonlinear T-duality transformations for massless R-R and NS-NS fields, when there is one Killing coordinate y, are:

$$\begin{aligned}
e^{2\widetilde{\phi}} &= \frac{e^{2\phi}}{G_{yy}} \\
\widetilde{G}_{yy} &= \frac{1}{G_{yy}} \\
\widetilde{G}_{\mu y} &= \frac{B_{\mu y}}{G_{yy}} \\
\widetilde{G}_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu y}G_{\nu y} - B_{\mu y}B_{\nu y}}{G_{yy}} \\
\widetilde{B}_{\mu y} &= \frac{G_{\mu y}}{G_{yy}} \\
\widetilde{B}_{\mu\nu} &= B_{\mu\nu} - \frac{B_{\mu y}G_{\nu y} - G_{\mu y}B_{\nu y}}{G_{yy}} \\
\tilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu y} &= \mathcal{C}^{(n-1)}_{\mu\cdots\nu} \\
\tilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu} &= \mathcal{C}^{(n+1)}_{\mu\cdots\nu y}
\end{aligned}$$

where $\mathcal{C} = e^{B}C$ and $\mu, \nu, \neq y$. In above transformation the metric is the string frame metric. If y is identified on a circle of radius R, *i.e.*, $y \sim y + 2\pi R$, then after T-duality the radius becomes $\tilde{R} = \alpha'/R$. The string coupling is also shifted as $\tilde{g} = g\sqrt{\alpha'}/R$.

We will consider perturbations around flat space where the metric takes the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is a small perturbation. We denote the Riemann tensor to linear order in h by $\mathcal{R}_{\mu\nu\rho\lambda}$. This linear Riemann tensor is,

$$\mathcal{R}_{\mu\nu\rho\lambda} \;=\; \frac{1}{2}(h_{\mu\lambda,\nu\rho} + h_{\nu\rho,\mu\lambda} - h_{\mu\rho,\nu\lambda} - h_{\nu\lambda,\mu\rho})$$

where as usual commas denote partial differentiation. Assuming the other NSNS fields are also small perturbations, the above transformations take the following linear form:

$$\begin{aligned}
&\tilde{\phi} = \phi - \frac{1}{2}h_{yy},\ \tilde{h}_{yy} = -h_{yy},\ \tilde{h}_{\mu y} = B_{\mu y},\ \tilde{B}_{\mu y} = h_{\mu y},\ \tilde{h}_{\mu\nu} = h_{\mu\nu},\ \tilde{B}_{\mu\nu} = B_{\mu\nu}\\
&\tilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu y} = \mathcal{C}^{(n-1)}_{\mu\cdots\nu},\ \ \tilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu} = \mathcal{C}^{(n+1)}_{\mu\cdots\nu y}
\end{aligned}$$

Our strategy for finding T-duality invariant couplings corresponding to a D-brane coupling is as follows: Suppose we are implementing T-duality along a world volume direction of D_p-brane denoted y. A world-volume index can be separated into y index and the world-volume index which does not include y, *i.e.*,

$$\mathcal{R}_{a\cdots i\cdots}\mathcal{R}^{a\cdots i\cdots} = \mathcal{R}_{\tilde{a}\cdots i\cdots}\mathcal{R}^{\tilde{a}\cdots i\cdots} + \mathcal{R}_{y\cdots i\cdots}\mathcal{R}^{y\cdots i\cdots}$$

Under T-duality the D_p-brane transforms to D_{p-1}-brane and the above coupling transforms to

$$\rightarrow \tilde{\mathcal{R}}_{a\cdots\tilde{i}\cdots}\tilde{\mathcal{R}}^{a\cdots\tilde{i}\cdots} + \tilde{\mathcal{R}}_{y\cdots\tilde{i}\cdots}\tilde{\mathcal{R}}^{y\cdots\tilde{i}\cdots}$$

The indices are not complete in the T-dual theory. One must add new couplings to the action to have the complete indices in the T-dual theory.

3 S-duality

The D_3-brane and the Einstein frame metric are invariant under $SL(2,Z)$, and the B-field and the RR two-form transform as doublet, *i.e.,*

$$\mathcal{B}^{(2)} = \begin{pmatrix} B^{(2)} \\ C^{(2)} \end{pmatrix}$$

Under a $SL(2,Z)$ transformation by

$$\Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

the $\mathcal{B}$-field transforms linearly by the rule

$$\mathcal{B} \to \Lambda\mathcal{B}$$

The axion and the dilaton combine into a complex scalar field $\tau = C + ie^{-\phi}$. This field transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

In this case also one should add new terms to the action to make it invariant under the above S-duality transformations.

4 DBI couplings

4.1 New couplings from T-duality

Let us consider each curvature terms separately.

4.1.1 $\mathcal{R}_{abcd}\mathcal{R}^{abcd}$ term

We begin by examining the curvature term $\mathcal{R}_{abcd}\mathcal{R}^{abcd}$ under linear T-duality transformation. We are implementing T-duality along a world volume direction of D_p-brane. So we write it as

$$(\mathcal{R}_{abcd})^2 \;=\; (\mathcal{R}_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}})^2 + (h_{\tilde{a}y,\tilde{b}\tilde{c}} - h_{\tilde{b}y,\tilde{a}\tilde{c}})^2 + (h_{yy,\tilde{a}\tilde{b}})^2$$

Our notation is such that *e.g.*, $(\mathcal{R}_{abcd})^2 = \mathcal{R}_{abcd}\mathcal{R}^{abcd}$. Under the linear T-duality transformation, it transforms to

$$(\mathcal{R}_{abcd})^2 \ \rightarrow \ (\mathcal{R}_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}})^2 + (B_{\tilde{a}y,\tilde{b}\tilde{c}} - B_{\tilde{b}y,\tilde{a}\tilde{c}})^2 + (h_{yy,\tilde{a}\tilde{b}})^2$$

Because there are incomplete transverse index y, one concludes that the original curvature term is not consistent with T-duality even in the absence of the B-field. One must add some terms to the curvature term to have completed indices in the T-dual theory. A T-dual expression can be constructed from graviton and B-field, or can be constructed from graviton, B-field and dilaton. In the former case, one may consider the following terms:

$$(\mathcal{R}_{abcd})^2 + (B_{ai,bc} - B_{bi,ac})^2 + \alpha(h_{ij,ab})^2 + \beta(B_{ci,ab})^2 + \gamma(h_{cd,ab})^2$$

where α, β, γ are three constants that we will find by requiring the above combination to be invariant under the T-duality transformation. Doing the same steps as we have done for $(\mathcal{R}_{abcd})^2$, one finds that the above expression transforms under the T-duality to the following terms:

$$(h_{yy,\tilde{a}\tilde{b}})^2(-1+\alpha-\beta+\gamma) + (h_{yi,\tilde{a}\tilde{a}})^2(2-2\alpha+\beta) + (B_{y\tilde{c},\tilde{a}\tilde{b}})^2(-\beta+2\gamma) + \cdots$$

where dots refer to the terms which have complete indices. To have a T-dual expression, one must impose the vanishing of the coefficient of the above three terms which have the incomplete index y. This fixes α, β and γ. The T-dual completion of the curvature term is then

$$(\mathcal{R}_{abcd})^2 + (B_{ai,bc} - B_{bi,ac})^2 - 2(B_{ci,ab})^2 - (h_{cd,ab})^2 \tag{3}$$

One may try to find a T-dual completion of $(\mathcal{R}_{abcd})^2$ by including the derivative of the dilaton field. We have found that it is very unlikely to have such a T-dual completion.

4.1.2 $\hat{\mathcal{R}}_{ab}\hat{\mathcal{R}}^{ab}$ term

The T-dual completion in this case is the following couplings:

$$(\hat{\mathcal{R}}_{ab} + \phi_{,ab})^2 + \frac{1}{2}(B_{ic,ca} - B_{ia,cc})^2 - \frac{1}{2}(B_{ia,cc})^2 - \frac{1}{4}(h_{ab,cc})^2 \tag{4}$$

4.1.3 $\mathcal{R}_{abij}\mathcal{R}^{abij}$ term

The T-dual completion in this case is the following couplings:

$$(\mathcal{R}_{abij})^2+\frac{1}{2}(B_{ki,aj}-B_{kj,ai})^2+\frac{1}{2}(B_{ac,bi}-B_{bc,ai})^2-\frac{1}{2}(B_{ki,aj})^2-\frac{1}{2}(B_{ac,bi})^2 \quad (5)$$

4.1.4 $\hat{\mathcal{R}}_{ij}\hat{\mathcal{R}}^{ij}$ term

The T-dual completion in this case is the following couplings:

$$(\hat{\mathcal{R}}_{ij}+\phi_{ij})^2+\frac{1}{2}(B_{ab,bj}-B_{aj,bb})^2+\frac{1}{4}(h_{ab,cc})^2 \quad (6)$$

4.1.5 $\partial H\partial H$ couplings

Now adding the equations (3), (4), (5), (6), and using the integration by part, one finds that the non-tensor graviton terms are canceled, *i.e.,* the particular combination of the curvature squared terms is invariant under the linear T-duality transformations in the absence of the B-field. In the presence of B-field, consistency with the T-duality transformations requires that one must include in the action the B-field couplings (3), (4), (5), (6). Ignoring some total derivative terms, one finds the following couplings:

$$\begin{aligned}&B_{ki,aj}B_{kj,ai}+B_{ac,bi}B_{bc,ai}-\frac{1}{2}(B_{ki,aj})^2-\frac{1}{2}(B_{ac,bi})^2\\&-(B_{ic,ca})^2-2B_{ab,bi}B_{ai,cc}+(B_{ab,bi})^2+(B_{ai,bb})^2\end{aligned}$$

We have checked that these couplings are reproduced by the $\mathcal{O}(\alpha'^2)$ contact terms of the corresponding disk-level scattering amplitude.

One can easily write the result in terms of H as

$$S_{DBI}\supset\int d^{p+1}x\, e^{-\phi}\sqrt{-G}\Big[\frac{1}{6}\partial_a H_{ijk}\partial^a H^{ijk}+\frac{1}{3}\partial_i H_{abc}\partial^i H^{abc}-\frac{1}{2}\partial_a H_{bci}\partial^a H^{bci}\Big]$$

where the indices are raised and lowered by the flat metrics η_{ab} and η_{ij}.

4.2 New couplings from S-duality

Since B-field and the RR 2-form potential transform as doublet under S-duality, there must be the following couplings for D_3-brane:

$$S_{DBI} \supset \int d^4x \, e^{\phi}\sqrt{-G}\Big[\frac{1}{6}F_{ijk,a}F^{ijk,a} + \frac{1}{3}F_{abc,i}F^{abc,i} - \frac{1}{2}F_{bci,a}F^{bci,a}\Big] \qquad (7)$$

We have checked that these couplings are reproduced by the $\mathcal{O}(\alpha'^2)$ contact terms of the corresponding disk-level scattering amplitude.

To study the transformation of the above couplings under S-duality, one should first rescale the metric from string frame to the Einstein frame $G_{\mu\nu} = e^{\phi/2}g_{\mu\nu}$. The B-field couplings are multiplied by $e^{-2\phi}$ and the dilaton drops out of the above RR couplings.

Using the matrix $\mathcal{M}$

$$\mathcal{M} = e^{\phi}\begin{pmatrix} |\tau|^2 & -C \\ -C & 1 \end{pmatrix} \qquad (8)$$

which transforms under the $SL(2,Z)$ as

$$\mathcal{M} \to (\Lambda^{-1})^T \mathcal{M}\Lambda^{-1} \qquad (9)$$

one can extend the couplings to the following form in the Einstein frame:

$$S_{DBI} \int d^4x \, e^{-\phi}\sqrt{-g}\Big[\frac{1}{6}\mathcal{F}^T_{ijk,a}\mathcal{M}\mathcal{F}^{ijk,a} + \frac{1}{3}\mathcal{F}^T_{abc,i}\mathcal{M}\mathcal{F}^{abc,i} - \frac{1}{2}\mathcal{F}^T_{bci,a}\mathcal{M}\mathcal{F}^{bci,a}\Big]$$

where $\mathcal{F} = d\mathcal{B}$. Apart from the overall dilaton factor $e^{-\phi}$, it is invariant under the $SL(2,Z)$ transformation. The above action include also new couplings $CC^{(2)}B^{(2)}$ and $CCB^{(2)}B^{(2)}$ which should be reproduced by the disk level S-matrix element of two RR and one or two NSNS vertex operators.

Consider the non-holomorphic Eisenstein series defined by

$$2\zeta(2s)E_s(\tau,\bar{\tau}) \;=\; \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

where $\tau = \tau_1 + i\tau_2$. It is invariant under the $SL(2,Z)$ transformation. For $s=1$, this series diverges logarithmically. The regularized function has the following expansion [1, 2]:

$$2\zeta(2)E_1(\tau,\bar{\tau}) = \zeta(2)\tau_2 - \frac{\pi}{2}\ln(\tau_2) + \pi\sqrt{\tau_2}\sum_{m\neq 0, n\neq 0}\left|\frac{m}{n}\right|^{1/2} K_{1/2}(2\pi|mn|\tau_2)e^{2\pi i m n \tau_1}$$

where the first term corresponds to $n=0$ in the series. This term is exactly the dilaton factor. So one may replace the factor $e^{-\phi}$ in the couplings by the $SL(,Z)$ invariant function E_1.

The evidence for this replacement is the following. The B-field couplings are related to the gravity couplings by T-duality. On the other hand the $SL(2,Z)$ invariant form of the gravity couplings has this nonperturbative factor [1, 2]. In fact, the regularized function (10) is proportional to $\log(\tau_2|\eta(\tau)|^4)$ [1] where $\eta(\tau)$ is the Dedekind η-function. The $\log\tau_2$ piece is nonanalytic and comes from the annulus [2] and the remaining part appears in the Wilsonian effective action found in [1]. Hence, one expects the $SL(2,Z)$ invariant form of the $\mathcal{B}$-field couplings to be

$$S_{DBI} \supset \int d^4x\, E_1(\tau,\bar{\tau})\sqrt{-g}\Big[\frac{1}{6}\mathcal{F}^T_{ijk,a}\mathcal{M}\mathcal{F}^{ijk,a} + \frac{1}{3}\mathcal{F}^T_{abc,i}\mathcal{M}\mathcal{F}^{abc,i} - \frac{1}{2}\mathcal{F}^T_{bci,a}\mathcal{M}\mathcal{F}^{bci,a}\Big]$$

The second term in the expansion of E_1 is the annulus contribution to the 1PI effective action. All other terms are the D-instanton contributions.

5 CS couplings

We consider the following $O(\alpha'^2)$ part of the CS couplings:

$$\frac{T_p}{2!2!(p-3)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}\Big[R_{ab}{}^{ef}R_{cdfe} - R_{ab}{}^{kl}R_{cdlk}\Big]$$

where we have employed the static gauge. This coupling at the linearized level is then

$$\frac{T_p}{2!(p-3)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}\Big[h_a{}^f{}_{,b}{}^e(h_{ce,df} - h_{cf,de}) - h_a{}^l{}_{,b}{}^k(h_{ck,dl} - h_{cl,dk})\Big] \quad (10)$$

The indices that are contracted with the volume form are totally antisymmetric so we do not use the antisymmetric notation for them. The above couplings have been verified by the S-matrix element of one RR and two graviton vertex operators.

5.1 New couplings from T-duality

We will examine the expression (10) under the linear T-duality transformations, and find its corresponding T-dual multiplet. We call this multiplet the Chern-Simons multiplet.

5.1.1 Chern-Simons multiplet

We begin by implementing T-duality along a world volume direction of D$_p$-brane denoted y. There are two cases to consider: First when y is carried by the RR potential $\mathcal{C}^{(p-3)}$ and second when y is carried by the metric. In the former case, we write (10) as

$$\frac{T_p}{2!(p-4)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-4}yabcd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-5}y}\Big[h_a{}^f{}_{,b}{}^e(h_{ce,df}-h_{cf,de}) \\ -h_a{}^l{}_{,b}{}^k(h_{ck,dl}-h_{cl,dk})\Big]$$

The indices e,f include the Killing coordinate y which is a world volume coordinate. However, in the T-dual theory y is a transverse coordinate. To be able to use the T-duality transformation rules, we separate y from e,f. Hence, we write the above equation as

$$\frac{T_p}{2!(p-4)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-4}yabcd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-5}y}\Big[h_a{}^{\tilde{f}}{}_{,b}{}^{\tilde{e}}(h_{c\tilde{e},d\tilde{f}}-h_{c\tilde{f},d\tilde{e}})-h_a{}^y{}_{,b}{}^{\tilde{e}}h_{cy,d\tilde{e}} \\ -h_a{}^l{}_{,b}{}^k(h_{ck,dl}-h_{cl,dk})\Big]$$

where the "tilde" over the world volume indices e,f means they do not include the Killing direction y. Now, the above equation transforms under

T-duality to the following couplings of D$_{p-1}$-brane:

$$\frac{T_{p-1}}{2!(p-4)!}\int d^px\epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{(p-4)}_{a_0\cdots a_{p-5}}\Big[h_a{}^f{}_{,b}{}^e(h_{ce,df}-h_{cf,de})-B_a{}^y{}_{,b}{}^eB_{cy,de}$$
$$-h_a{}^{\tilde{l}}{}_{,b}{}^{\tilde{k}}(h_{c\tilde{k},d\tilde{l}}-h_{c\tilde{l},d\tilde{k}})\Big]$$

where we have used the fact that $T_p\sim 1/g_s$ and the relation $2\pi\sqrt{\alpha'}T_p=T_{p-1}$. In above equation the "tilde" over the transverse indices k,l means they do not include the Killing direction y which is now a direction normal to the D$_{p-1}$-brane. The contracted indices of the second and third terms are not complete, *i.e.,* the second term has index y which does not include all other transverse coordinates, and the last term has index $\tilde{l}$ which does not include the transverse coordinate y. This indicates that the original action (10) is not consistent with the linear T-duality.

To remedy this failure, one has to add some new couplings. These couplings must be such that when they are combined with the couplings (10), their indices remain complete after transforming them under the linear T-duality transformations (3). Consider the following couplings on the world volume of the D$_p$-brane:

$$\frac{T_p}{2!(p-3)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}\Big[-B_a{}^k{}_{,b}{}^eB_{ck,de}+B_a{}^e{}_{,b}{}^kB_{ce,dk}\Big]\qquad(11)$$

Doing the same steps as we have done for the couplings (10), one finds that the above couplings transforms to the following couplings of D$_{p-1}$-brane:

$$\frac{T_{p-1}}{2!(p-4)!}\int d^px\epsilon^{a_0\cdots a_{p-5}abcd}\mathcal{C}^{(p-4)}_{a_0\cdots a_{p-5}}\Big[-B_a{}^{\tilde{k}}{}_{,b}{}^eB_{c\tilde{k},d\tilde{e}}+B_a{}^e{}_{,b}{}^kB_{ce,dk}+h_a{}^y{}_{,b}{}^kh_{cy,dk}\Big]$$

In this equation also the index $\tilde{k}$ in the first and the index y in the last terms are not complete. This indicates that the coupling (11) is not consistent with the T-duality either. However, the combination of actions (10) and (11) are consistent with T-duality when y is an index on the RR potential. That is, the following couplings of D$_p$-brane:

$$\frac{T_p}{2!(p-3)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}\Big[h_a{}^f{}_{,b}{}^e(h_{ce,df}-h_{cf,de})-B_a{}^k{}_{,b}{}^eB_{ck,de}$$
$$-h_a{}^l{}_{,b}{}^k(h_{ck,dl}-h_{cl,dk})+B_a{}^e{}_{,b}{}^kB_{ce,dk}\Big]\qquad(12)$$

are consistent with the linear T-duality transformations when the RR potential carries the Killing index.

There are two possibilities for the metric/B-field terms in (12) to carry the Killing coordinate y, *i.e.*, either a or c carries the index y. One can write the D_p-brane couplings (12) as

$$\frac{T_p}{(p-3)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-4}abyd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}\Big[h_a{}^f{}_{,b}{}^e(h_{ye,df}-h_{yf,de})-B_a{}^k{}_{,b}{}^eB_{yk,de}$$
$$-h_a{}^l{}_{,b}{}^k(h_{yk,dl}-h_{yl,dk})+B_a{}^e{}_{,b}{}^kB_{ye,dk}\Big]$$

Doing the same steps as we have done above, one finds the transformation of the above couplings under T-duality gives the following couplings for D_{p-1}-brane:

$$\frac{T_{p-1}}{(p-3)!}\int d^px\epsilon^{a_0\cdots a_{p-4}abd}\mathcal{C}^{(p-2)}_{a_0\cdots a_{p-4}}{}^y\Big[-h_a{}^f{}_{,b}{}^e(B_{ye,df}-B_{yf,de})+B_a{}^k{}_{,b}{}^eh_{yk,de}$$
$$+h_a{}^l{}_{,b}{}^k(B_{yk,dl}-B_{yl,dk})-B_a{}^e{}_{,b}{}^kh_{ye,dk}\Big]\tag{13}$$

In this case the world volume indices e,f and the transverse indices k,l are all complete. However, the y index is not a complete index. Inspired by the above couplings, one may guess that there should be the following couplings for the D_p-brane:

$$\frac{T_p}{(p-2)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-3}abd}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^i\Big[-h_a{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})+B_a{}^k{}_{,b}{}^eh_{ik,de}$$
$$+h_a{}^l{}_{,b}{}^k(B_{ik,dl}-B_{il,dk})-B_a{}^e{}_{,b}{}^kh_{ie,dk}\Big]\tag{14}$$

One can easily verify that the above couplings are invariant under the linear T-duality transformations (3) when the world volume Killing coordinate y is carried by the RR potential. The RR potential $\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-4}y}{}^i$ transforms to $\mathcal{C}^{(p-2)}_{a_0\cdots a_{p-4}}{}^i$ in which the transverse index i does not include y, and the couplings for $i=y$ are given by (13).

In order to proceed further, one observes that there is one possibility for the metric/B-field terms in (14) to carry the T-dual coordinate y, *i.e.*, a

carries the index y. One can write it as

$$\frac{T_p}{(p-2)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-3}ybd}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}\Big[-h_y{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})+B_y{}^k{}_{,b}{}^e h_{ik,de}$$
$$+h_y{}^l{}_{,b}{}^k(B_{ik,dl}-B_{il,dk})-B_y{}^e{}_{,b}{}^k h_{ie,dk}\Big]$$

Under T-duality it transforms to the following couplings for D$_{p-1}$-brane:

$$\frac{T_{p-1}}{(p-2)!}\int d^{p}x\epsilon^{a_0\cdots a_{p-3}bd}\mathcal{C}^{(p)}_{a_0\cdots a_{p-3}}{}^{iy}\Big[B_y{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})-h_y{}^k{}_{,b}{}^e h_{ik,de}$$
$$-B_y{}^l{}_{,b}{}^k(B_{ik,dl}-B_{il,dk})+h_y{}^e{}_{,b}{}^k h_{ie,dk}\Big]$$

where now again the world volume indices e, f and the transverse indices k, l are all complete, whereas the y index is not a complete index. Inspired by the above couplings, one finds the following couplings for the D$_p$-brane:

$$\frac{T_p}{2!(p-1)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-2}bd}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}\Big[B_j{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})-h_j{}^k{}_{,b}{}^e h_{ik,de}$$
$$-B_j{}^l{}_{,b}{}^k(B_{ik,dl}-B_{il,dk})+h_j{}^e{}_{,b}{}^k h_{ie,dk}\Big] \quad (15)$$

There is no index in the B-field/metric in (15) that contracts with the volume form. Hence, the combination of couplings (12), (14) and (15) forms a complete T-dual multiplet, *i.e.,* the CS multiplet. This multiplet is

$$T_p\int d^{p+1}x\left(\frac{\epsilon^{a_0\cdots a_{p-4}abcd}}{2!(p-3)!}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}[h_a{}^f{}_{,b}{}^e(h_{ce,df}-h_{cf,de})-B_a{}^k{}_{,b}{}^e B_{ck,de}]\right.$$
$$+\frac{\epsilon^{a_0\cdots a_{p-3}abd}}{(p-2)!}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}[-h_a{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})+B_a{}^k{}_{,b}{}^e h_{ik,de}]$$
$$\left.+\frac{\epsilon^{a_0\cdots a_{p-2}bd}}{2!(p-1)!}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}[B_j{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})-h_j{}^k{}_{,b}{}^e h_{ik,de}]\right) \quad (16)$$

There are similar expressions as above with opposite sign and with replacing the world volume indices (e, f) by the transverse indices (k, l) and replacing (e, k) by (k, e). We call the $\mathcal{C}^{(p-3)}$ couplings as the first component of the multiplet, the $\mathcal{C}^{(p-1)}$ couplings as the second component of the multiplet, and so on.

5.1.2 $\mathcal{C}^{(p-3)}$ couplings

One can easily check that the first component of the CS multiplet (16) is not invariant under B-field gauge transformation. To write it in terms of field strength H, we add another T-dual multiplet to (16). Since the gravity couplings to $\mathcal{C}^{(p-3)}$ are those given by (16), the first component of the T-dual multiplet must include only B-field. Consider the following couplings for $\mathcal{C}^{(p-3)}$:

$$\frac{T_p}{2!(p-3)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}(B_{ak,be}-B_{ae,bk})B_{cd}{}^{,ek} \tag{17}$$

It is easy to verify that this coupling is invariant under linear T-duality transformations (3) when the Killing coordinate y is carried by the RR potential. Doing the same steps as we have done in the previous section, one finds the T-dual multiplet corresponding to (17) to be

$$\begin{aligned} T_p\int d^{p+1}x\Bigg(&\frac{\epsilon^{a_0\cdots a_{p-4}abcd}}{2!(p-3)!}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}B_{ak,be}B_{cd}{}^{,ek}\\ &+\frac{\epsilon^{a_0\cdots a_{p-3}abd}}{2!(p-2)!}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}[h_{ik,be}B_{ad}{}^{,ek}-2B_{ak,be}h_{id}{}^{,ek}]\\ &+\frac{\epsilon^{a_0\cdots a_{p-2}bd}}{(p-1)!}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}[-h_{ik,be}h_{jd}{}^{,ek}+\frac{1}{2}B_{dk,be}B_{ij}{}^{,ek}]\\ &+\frac{\epsilon^{a_0\cdots a_{p-1}b}}{2!p!}\mathcal{C}^{(p+3)}_{a_0\cdots a_{p-1}}{}^{ijn}h_{ie,bk}B_{jn}{}^{,ek}\Bigg) \end{aligned} \tag{18}$$

There are similar expressions as above with opposite sign and with replacing (e,k) by (k,e).

Now, the first components of the CS multiplet (16) and the above multiplet can be written in terms of H, *i.e.,*

$$\begin{aligned} \frac{T_p}{2!2!(p-3)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{p-3}_{a_0\cdots a_{p-4}}[&\mathcal{R}_{ab}{}^{ef}\mathcal{R}_{cdfe}-\mathcal{R}_{ab}{}^{kl}\mathcal{R}_{cdlk}\\ &-\frac{1}{2}H_{abk,e}H_{cd}{}^{k,e}+\frac{1}{2}H_{abe,k}H_{cd}{}^{e,k}] \end{aligned} \tag{19}$$

where $H_{\mu\nu\rho}=B_{\mu\nu,\rho}+B_{\rho\mu,\nu}+B_{\nu\rho,\mu}$. The terms in the second line are reproduced by the S-matrix calculation [3]. Unlike the gravity couplings in the first line, the B-field couplings in the second line are not invariant under the RR gauge transformation.

One may then expect that there might be some other T-dual multiplets that should be added to the above couplings. As we have pointed out above, their first components must include only B-field. The presence of such couplings can be fixed by the S-matrix calculation. In fact the couplings (19) as well as the following couplings are produced by the S-matrix element of one RR and two NSNS vertex operators [3]:

$$\mathcal{C}^{(p-3)}_{a_4\cdots a_p}\bigg(\frac{1}{2!2!}H^{ea_0a_1}{}_{,ef}H^{fa_2a_3}+\frac{1}{3!}H^{a_0a_1a_2}{}_{,ek}H^{kea_3} \\ +\frac{1}{2!2!}H^{a_0a_1e,k}{}_{e}H^{a_2a_3}{}_{k}+\frac{1}{3!}H^{a_0a_1a_2}{}_{,e}H^{efa_3}{}_{,f}+\frac{1}{3!}H^{a_0a_1a_2}{}_{,k}H^{kea_3}{}_{,e}\bigg) \tag{20}$$

The S-matrix element produces also some massless open string poles at order $O(\alpha'^2)$ which are reproduced by the CS action at order $O(\alpha'^0)$ and the following couplings [3]:

$$-\bigg(\frac{1}{2!2!}C^{(p-3)}_{a_4\cdots a_p,k}(2H_{a_0a_1}{}^{e,k}{}_{e}-H_{a_0a_1}{}^{k,e}{}_{e})(B_{a_2a_3}+2\pi\alpha' f_{a_2a_3}) \\ +\mathcal{C}^{(p-3)}_{a_4\cdots a_p}[\frac{1}{3!}H^{a_0a_1a_2,e}{}_{ef}(B^{fa_3}+2\pi\alpha' f^{fa_3})+\frac{1}{2!2!}H^{a_0a_1f,e}{}_{e}(B^{a_2a_3}+2\pi\alpha' f^{a_2a_3})_{,f}]\bigg) \tag{21}$$

as well as some massless closed string poles at this order. It is shown in [3] that even though the contact terms and the massless poles are not separately invariant under the RR gauge transformations, the combination of them are.

5.1.3 $\mathcal{C}^{(p-1)}$ couplings

It has been shown in [3] that the S-matrix element of one RR potential $C^{(p+5)}$ and two gravitons is zero, and the S-matrix element of one $C^{(p+5)}$ and two B-fields has only massless closed string poles at order $O(\alpha'^2)$. Moreover, one finds that the two S-matrix elements are exactly consistent with the linear T-duality. However, using the same steps as we have done in section 5.1.1, one finds that the couplings (19) have no $C^{(p+5)}$ component, and the couplings (20) and (21) have such component as

$$C^{(p+5)}_{a_0\cdots a_p}{}^{ijmn}B^{ij,e}{}_{ek}B^{mn,k}-C^{(p+5)}_{a_0\cdots a_p}{}^{ijmn,k}B^{ij,e}{}_{ek}B^{mn}$$

That is, the contact-term T-dual multiplets have $C^{(p+5)}$ component whereas the S-matrix element does not have such contact term! This indicates that the above contact terms must be canceled with the $C^{(p+5)}$ component of the massless-pole T-dual multiplets.

This phenomenon may happen only when two derivatives in a coupling contract with each other. Or in momentum space a contact term is proportional to a Mandelstam variable. To clarify this point consider a contact term in S-matrix which is proportional to the Mandelstam variable $p_2\!\cdot\! V\!\cdot\! p_3$ where matrix V is the world volume metric, *i.e.,* $(\cdots)p_2\cdot V\cdot p_3 J$ where J is a function of the Mandelstam variables which has contact term at low energy and has no massless pole at , say, $p_1\cdot p_3$, and $(\cdots)$ includes the polarization tensors and second power of momenta. This term in $C^{(p-3)}$ component appear as contact term at low energy. Using the relations between these functions, the other components of this multiplet may be written as $(\cdots)p_2\!\cdot\! V\!\cdot\! p_3 J = [(\cdots)p_1\!\cdot\! p_2]I$ where I is another function which has massless pole at $p_1\cdot p_3$. The $(\cdots)p_1\cdot p_2$ part in the second hand side which is of order $O(\alpha'^2)$, may be combined with the other massless poles of the S-matrix element to produce a covariant/gauge invariant pole at order $O(\alpha'^2)$. Note that one can not do such a rearrangement for the contact terms at order $O(\alpha'^2)$ which are not proportional to a Mandelstam variable . They must be combined with the other contact-term T-dual multiplets of the S-matrix element to produce covariant/gauge invariant result. Hence, we will discuss the covariant/gauge invariant form of the terms in the contact-term T-dual multiplets which are not proportional to a Mandelstam variable.

One can easily verify that the structure of the couplings (20) and (21) is different from the couplings in the CS multiplet (16). In particular, the couplings (16) are antisymmetric under $(e,f)\rightarrow(k,e)$ and $(e,k)\rightarrow(k,l)$ whereas the couplings (20) and (21) do not have such symmetry. So they can not be combined to produce covariant/gauge invariant multiplets. For example, the $\mathcal{C}^{(p-1)}$ component of the multiplets (20) involves the metric in the form $h_{i\cdots,\cdots}$. Such structures can not be combined with the first term in the second line of (16) to make it invariant under the B-field gauge transformation. On the other hand, this term in momentum space is not proportional to a Mandelstam variable, so it can not be related to massless poles of the S-matrix element. These indicate that there must be other T-dual multiplets to make it invariant. The first component of these multiplets should be $\mathcal{C}^{(p-1)}$. The strategy for finding these multiplet is to find its first component by requiring them to be combined with the corresponding coupling in (16) to make it invariant under the B-field gauge transformation, and then using the same steps as in section 5.1.1 to find all the components of the T-dual multiplet.

There are two multiplets to do this job. The first multiplet which has only two components, is given by the following expression:

$$\alpha T_p\int d^{p+1}x\left(\frac{\epsilon^{a_0\cdots a_{p-3}abd}}{(p-2)!}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}[-h_a{}^f{}_{,b}{}^eB_{ef,id}-B_a{}^k{}_{,b}{}^eh_{ek,id}]\right.$$
$$\left.+\frac{\epsilon^{a_0\cdots a_{p-2}bd}}{(p-1)!}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}[B_j{}^f{}_{,b}{}^eB_{ef,id}+h_j{}^k{}_{,b}{}^eh_{ek,id}]\right)\tag{22}$$

There are similar expressions as above with opposite sign and with replacing the world volume indices (e,f) by the transverse indices (k,l) and replacing (e,k) by (k,e). The coefficient α is a constant that we will fix it shortly. One can easily check that the sum of the first term above for $\alpha=1$ and the first term in the second line of (16) can be written in terms of H, *i.e.*, $\mathcal{R}_{ab}{}^{ef}H_{ief,d}$. However, the sum of the first term in the second line above for $\alpha=1$ and the first terms in the third line of (16) can not be written in gauge invariant form.

The other possibility is the following multiplet:

$$\beta T_p\int d^{p+1}x\left(\frac{\epsilon^{a_0\cdots a_{p-3}abd}}{(p-2)!}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}[-(h_a{}^f{}_{,b}{}^e-h_a{}^e{}_{,b}{}^f)B_{ed,if}-B_a{}^k{}_{,b}{}^eh_{dk,ie}]\right.$$
$$+\frac{\epsilon^{a_0\cdots a_{p-2}bd}}{(p-1)!}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}[(B_j{}^f{}_{,b}{}^e-B_j{}^e{}_{,b}{}^f)B_{ed,if}-B_d{}^k{}_{,b}{}^eB_{jk,ie}$$
$$-(h_b{}^f{}_{,d}{}^e-h_b{}^e{}_{,d}{}^f)h_{ej,if}+h_j{}^k{}_{,b}{}^eh_{dk,ie}]$$
$$\left.+\frac{\epsilon^{a_0\cdots a_{p-1}b}}{p!}\mathcal{C}^{(p+3)}_{a_0\cdots a_{p-1}}{}^{ijn}[(B_j{}^f{}_{,b}{}^e-B_j{}^e{}_{,b}{}^f)h_{en,if}+h_n{}^k{}_{,b}{}^eB_{jk,ie}]\right)\tag{23}$$

Again, there are similar expressions as above with opposite sign and with replacing the world volume indices (e,f) by the transverse indices (k,l) and replacing (e,k) by (k,e). The coefficient β is a constant. One can easily check that the sum of the first term above for $\beta=1$ and the first term in the second line of (16) can be written in terms of H, *i.e.*, $\mathcal{R}_{ab}{}^{ef}H_{ide,f}$. In this case also the sum of the first term in the second line above for $\beta=1$ and the first term in the third line of (16) can not be written in gauge invariant form. To remedy the above failure, we will consider both multiplets with

$$\alpha=\beta=1/2$$

We will see in the next section that the above choice of the constants makes it possible to write the first term in the third line of (16) in gauge invariant form.

The last term in the first line of (23) is proportional to the Mandelstam variable $p_2\!\cdot\! V\!\cdot\! p_3$. So it may be combined with the other massless poles of the S-matrix element to be written as covariant/gauge invariant massless poles. However, the last term in the first line of (22) is not proportional to a Mandelstam variable in momentum space, so there must be other T-dual multiplets to make it covariant/gauge invariant. One can write it in covariant form by adding the terms $B_a{}^k{}_{,b}{}^e(h_{id,ek}-h_{ik,de}-h_{de,ik})$. The first two terms are proportional to the Mandelstam variable $p_2\!\cdot\! V\!\cdot\! p_3$ so we are not interested in their T-dual multiplets, and the last term is the $C^{(p-1)}$ component of the following multiplet:

$$\alpha T_p\int d^{p+1}x\left(\frac{\epsilon^{a_0\cdots a_{p-3}abd}}{(p-2)!}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}[B_a{}^k{}_{,b}{}^e h_{de,ik}]\right.$$
$$+\frac{\epsilon^{a_0\cdots a_{p-2}bd}}{(p-1)!}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}[-h_j{}^k{}_{,b}{}^e h_{de,ik}]$$
$$\left.+\frac{\epsilon^{a_0\cdots a_{p-1}b}}{p!}\mathcal{C}^{(p+3)}_{a_0\cdots a_{p-1}}{}^{ijn}[h_j{}^k{}_{,b}{}^e B_{ne,ik}]\right)\tag{24}$$

Again, there are similar expressions as above with opposite sign and with replacing (e,k) by (k,e).

Now, the $\mathcal{C}^{(p-1)}$ component of the sum of the above multiplet and the multiplets (23), (22) and (16) is

$$\frac{T_p}{(p-2)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-3}abd}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}\Big[-\frac{1}{2}\mathcal{R}_{ab}{}^{ef}(H_{ied,f}+\frac{1}{2}H_{ief,d})-\mathcal{R}_{iekd}H_{ab}{}^{k,e}$$
$$+\frac{1}{2}\mathcal{R}_{ab}{}^{kl}(H_{ikd,l}+\frac{1}{2}H_{ikl,d})+\mathcal{R}_{iked}H_{ab}{}^{e,k}\Big]$$

where we have added some terms which are proportional to the Mandelstam variables. They are related to the massless-pole T-dual multiplets in which we are not interested in this paper.

5.1.4 $\mathcal{C}^{(p+1)}$ couplings

Adding the multiplets (22) and (23) to the CS multiplet (16), one finds that the B-field terms in the $\mathcal{C}^{(p+1)}$ component of the CS multiplet can be written in terms of field strength H provided that one adds also the following multiplet:

$$\frac{1}{2}T_p\int d^{p+1}x\left(\frac{\epsilon^{a_0\cdots a_{p-2}bd}}{(p-1)!}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}[B^{fe}{}_{,jb}B_{ed,if}-h^{ek}{}_{,jb}h_{kd,ie}]\right.$$
$$\left.+\frac{\epsilon^{a_0\cdots a_{p-1}b}}{p!}\mathcal{C}^{(p+3)}_{a_0\cdots a_{p-1}}{}^{ijn}[B^{fe}{}_{,jb}h_{en,if}-h^{ek}{}_{,jb}B_{kn,ie}]\right) \tag{25}$$

There are similar expressions as above with opposite sign and with replacing the world volume indices (e,f) by the transverse indices (k,l) and replacing (e,k) by (k,e). The $\mathcal{C}^{(p+1)}$ component of the multiplets (16), (22), (23), (24), and (25) then become

$$\frac{T_p}{(p-1)!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-2}bd}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}\Big[\frac{1}{2}H_j{}^{fe}{}_{,b}H_{ied,f}+\frac{1}{4}\mathcal{R}_{bd}{}^{ef}\mathcal{R}_{ijfe}+\mathcal{R}^e{}_{jb}{}^k\mathcal{R}_{eidk}$$
$$-\frac{1}{2}H_j{}^{lk}{}_{,b}H_{ikd,l}-\frac{1}{4}\mathcal{R}_{bd}{}^{kl}\mathcal{R}_{ijlk}-\mathcal{R}^k{}_{jb}{}^e\mathcal{R}_{kide}\Big]$$

where again we have added some terms which are proportional to the Mandelstam variables. As we argued before, they may be resulted from the massless-pole T-dual multiplets. One may also add $H_{bd}{}^{k,e}H_{ijk,e}-H_{bd}{}^{e,k}H_{ije,k}$

to the above bracket. Since this contact term is proportional to the Mandelstam variables our present calculation which does not consider the massless-pole T-dual multiplets, can not fix the presence of this coupling. This term is very much line the coupling in the second line of (19). So it is very likely that such coupling is reproduced by the S-matrix calculation.

5.1.5 $\mathcal{C}^{(p+3)}$ couplings

The CS multiplet (16) does not have $C^{(p+3)}$ component. However, the multiplets (23), (24), and (25) which make the CS multiplet covariant/gauge invariant have such component. They combined to the following covariant/gauge invariant result:

$$\frac{T_p}{p!}\int d^{p+1}x\epsilon^{a_0\cdots a_{p-1}b}\mathcal{C}^{(p+3)}_{a_0\cdots a_{p-1}}{}^{ijn}\Big[\frac{1}{4}H_j{}^{fe}{}_{,b}\mathcal{R}_{nife}-\frac{1}{4}H_{ni}{}^{e,k}\mathcal{R}_{jebk}$$
$$-\frac{1}{4}H_j{}^{kl}{}_{,b}\mathcal{R}_{nikl}+\frac{1}{4}H_{ni}{}^{k,e}\mathcal{R}_{jkbe}\Big]$$

There are no coupling for $C^{(p+5)}$ which is consistent with the S-matrix calculation [3].

5.2 New couplings from S-duality

The couplings (19) for the self-dual D_3-brane are

$$S_{CS}\supset-\int d^4x\epsilon^{a_0\cdots a_3}C\Big[H_{a_0a_1a,i}H_{a_2a_3}{}^{a,i}-H_{a_0a_1i,a}H_{a_2a_3}{}^{i,a}\Big]$$

The above couplings have been also confirmed with scattering calculation [3]. The compatibility of these couplings with S-duality indicates that the disk level S-matrix element of three RR vertex operators produces the following couplings:

$$S_{CS}\supset-\int d^4x\epsilon^{a_0\cdots a_3}e^{2\phi}C\Big[F_{a_0a_1a,i}F_{a_2a_3}{}^{a,i}-F_{a_0a_1i,a}F_{a_2a_3}{}^{i,a}\Big]$$

where we have also used the standard rescaling $C\to e^{\phi}C$. In the Einstein frame the terms in the brackets can be combined into the $SL(2,Z)$ invariant form of $\mathcal{F}^T\mathcal{M}\mathcal{F}$. There is no overall dilaton factor $e^{-\phi}$. However, it has the RR scalar as the overall factor, *i.e.,*

$$S_{CS} \supset \int d^4x\epsilon^{a_0\cdots a_3}C\Big[\mathcal{F}^T_{a_0a_1a,i}\mathcal{M}\mathcal{F}_{a_2a_3}{}^{a,i} - \mathcal{F}^T_{a_0a_1i,a}\mathcal{M}\mathcal{F}_{a_2a_3}{}^{i,a}\Big] \tag{26}$$

Unlike the DBI case where the couplings in the Einstein frame have the dilaton factor, the above coupling has no such factor. The S-dual invariant of this coupling then does not include the non-holomorphic Eisenstein function. It should include another modular function $f(\tau,\bar{\tau})$. This function should have the following weak-expansion:

$$f(\tau,\bar{\tau}) \;\sim\; \tau_1 + \cdots$$

where dots stands for loops and D-instanton effects.

It has been shown in [1] that the gravity couplings in the effective Wilsonian CS theory for trivial normal bundle can be written in S-dual invariant form. The modular function that appears in this case is $\log(\eta(\tau)/\eta(\bar{\tau}))$. This function has the following weak-expansion [1]:

$$\log\frac{\eta(\tau)}{\eta(\bar{\tau})} \;=\; \frac{i\pi}{6}\tau_1 - \Big[q + \frac{3}{2}q^2 + \frac{4}{3}q^3 + \cdots - cc\Big]$$

where $q = e^{2\pi i\tau}$. The first term arises from the disk level amplitude and the series of the power q stand for the D-instanton corrections. The annulus effect is absent in the above function as it does not contribute to the Wilsonian effective action. The annulus effect in the 1PI effective action should make this function to be $SL(2,Z)$ invariant. So we expect the $SL(2,Z)$ invariant function $f(\tau,\bar{\tau})$ to be

$$f(\tau,\bar{\tau}) \;=\; \log\frac{A(\tau)\eta(\tau)}{A(\bar{\tau})\eta(\bar{\tau})} \tag{27}$$

where $A(\tau)$ is the annulus effect which makes $f(\tau,\bar{\tau})$ to be $SL(2,Z)$ invariant. In terms of this function, (26) can be written in the following S-dual form:

$$S_{CS} \supset \int d^4x\epsilon^{a_0\cdots a_3}f(\tau,\bar{\tau})\Big[\mathcal{F}^T_{a_0a_1a,i}\mathcal{M}\mathcal{F}_{a_2a_3}{}^{a,i} - \mathcal{F}^T_{a_0a_1i,a}\mathcal{M}\mathcal{F}_{a_2a_3}{}^{i,a}\Big]$$

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