Ageing phenomena far from equilibrium and local dynamical symmetries

Malte Henkel

Groupe de Physique Statistique,
Départment de Physique de la Matière et des Matériaux,
Institut Jean Lamour, CNRS – Nancy Université, France

collaborators:

R. Cherniha, S.B. Dutta, X. Durang, J.-Y. Fortin, M. Pleimling, S. Stoimenov, J. Unterberger

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Contents :

I. Ageing phenomena and dynamical scaling
physical ageing ; scaling behaviour and exponents ; mean-field theory

II. Local scaling with $z = 2$
Schrödinger and ageing algebras ; dynamical symmetry of the heat equation ; parabolic sub-algebras and dualisation ; stochastic field-theory ; computation of response functions ; tests of LSI

III. Local scaling with $z \neq 2$
Axioms of LSI ; Classification of 'mass-less' case ; Construction of mass terms ; Link to factorisable scattering ? ; computation of correlation functions ($z = 2$) ; tests

IV. Recent extensions

Conclusions

MH & PLEIMLING, *Non-equilibrium phase transitions, vol. 2* (2010)

I. Ageing phenomena and dynamical scaling

Equilibrium critical phenomena : **scale-invariance**
For sufficiently **local** interactions : extend to conformal invariance
space-dependent re-scaling (angles conserved) $\mathbf{r} \mapsto \mathbf{r}/b(\mathbf{r})$ POLYAKOV 70

In **two** dimensions : ∞ many conformal transformations
($w \mapsto f(w)$ analytic)
$\Rightarrow$ exact predictions for critical exponents, correlators, ... BPZ 84

What about **time**-dependent critical phenomena ?
Characterised by **dynamical exponent** z : $t \mapsto tb^{-z}$, $\mathbf{r} \mapsto \mathbf{r}b^{-1}$

Can one extend to **local** dynamical scaling, with $z \neq 1$?
If $z = 2$, the **Schrödinger group** is an example : JACOBI 1842, LIE 1881

$$t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta} \;,\; \mathbf{r} \mapsto \frac{\mathcal{R}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta} \;;\; \alpha\delta - \beta\gamma = 1$$

$\Rightarrow$ study **ageing** phenomena as paradigmatic example

Ageing phenomena

why do materials look old after some time ?

known & practically used since prehistoric times (metals, glasses)
systematically studied in physics since the 1970s STRUIK '78
occur in widely different systems

(structural glasses, spin glasses, polymers, simple magnets, . . .)

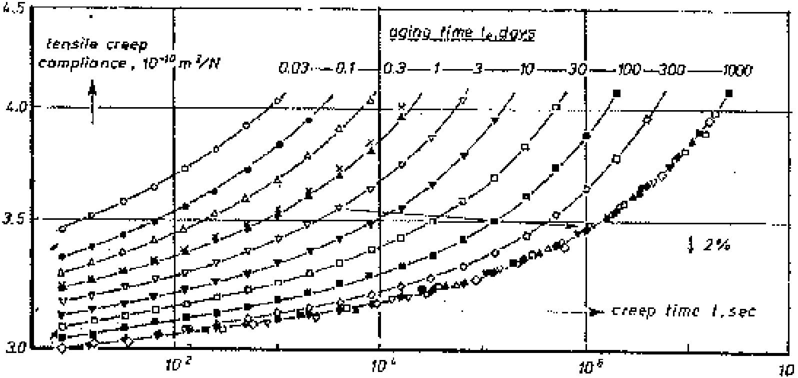
The three defining properties of **ageing** :

1. slow relaxation (non-exponential !)
2. **no** time-translation-invariance (TTI)
3. dynamical scaling

'Magnets' : **no** disorder, **no** frustration $\longrightarrow$ more simple to understand

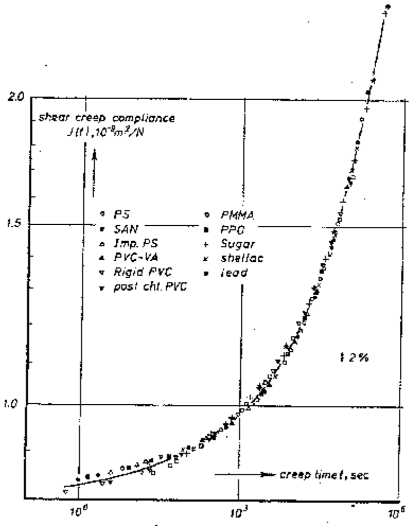
Question : what is the current evidence for larger,

local scaling symmetries ?



1. observe slow relaxation after quenching PVC from melt to low T
2. creep curves depend on **waiting time** t_e and **creep time** t
3. find master curve for all (t, t_e) $\longrightarrow$ dynamical scaling

$\rightarrow$ three defining properties of **physical ageing**



master curves of distinct materials are **identical**

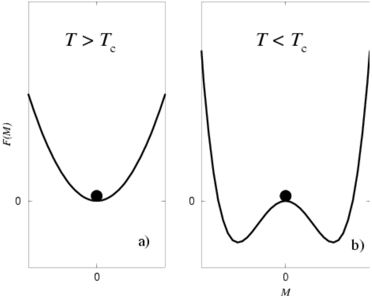
$\longrightarrow$ Universality !

good for theorists . . .

STRUIK 78

consider a simple magnet (ferromagnet, i.e. Ising model)

1. prepare system initially at high temperature $T \gg T_c > 0$
2. quench to temperature $T < T_c$ (or $T = T_c$)
   $\rightarrow$ non-equilibrium state
3. fix T and observe dynamics



competition :

at least 2 equivalent ground states

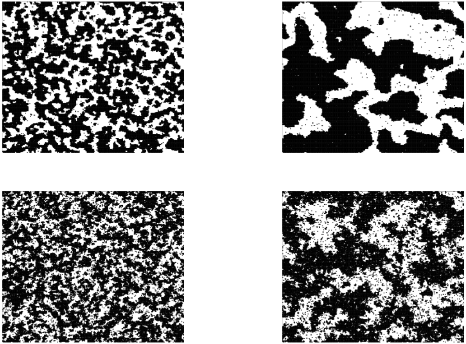
local fields lead to rapid local ordering

no global order, relaxation time ∞

formation of ordered domains, of linear size $L = L(t) \sim t^{1/z}$

dynamical exponent z

$t = t_1$ $t = t_2 > t_1$



magnet $T < T_c$

$\longrightarrow$ordered cluster

magnet $T = T_c$

$\longrightarrow$correlated cluster

growth of ordered/correlated domains, of typical linear size

$$L(t) \sim t^{1/z}$$

dynamical exponent z : determined by equilibrium state

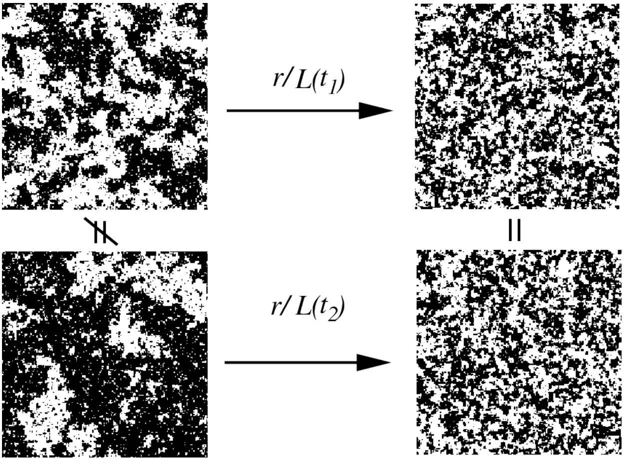
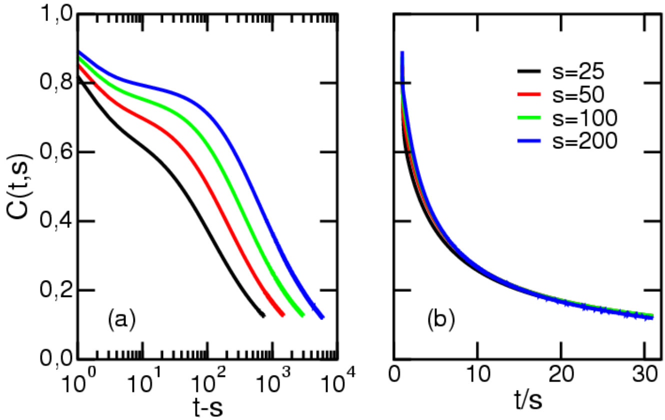


illustration of statistical self-similarity for different times $t_1 < t_2$

Walter '10

Dynamical scaling in the ageing $3D$ Ising model, $T < T_c$



no time-translation invariance — dynamical scaling

$C(t,s)$: autocorrelation function, quenched to $T < T_c$

scaling regime : $t, s \gg \tau_{\rm micro}$ and $t - s \gg \tau_{\rm micro}$

Question : derive scaling function in a model-independent way ?

Two-time observables

time-dependent order-parameter $\phi(t, \mathbf{r})$

two-time **correlator** $\quad C(t,s) := \langle \phi(t,\mathbf{r})\phi(s,\mathbf{r})\rangle - \langle \phi(t,\mathbf{r})\rangle \langle \phi(s,\mathbf{r})\rangle$

two-time **response** $\quad R(t,s) := \left. \frac{\delta \langle \phi(t,\mathbf{r})\rangle}{\delta h(s,\mathbf{r})} \right|_{h=0} = \left\langle \phi(t,\mathbf{r})\widetilde{\phi}(s,\mathbf{r}) \right\rangle$

t : observation time, s : waiting time

Scaling regime : $t, s \gg \tau_{\rm micro}$ **and** $t - s \gg \tau_{\rm micro}$

$$C(t,s) = s^{-b} f_C\left(\frac{t}{s}\right) \;,\; R(t,s) = s^{-1-a} f_R\left(\frac{t}{s}\right)$$

asymptotics : $f_{C,R}(y) \sim y^{-\lambda_{C,R}/z}$ for $y \gg 1$

λ_C : autocorrelation exponent, λ_R : autoresponse exponent,
z : dynamical exponent, a, b : ageing exponents

How to understand these scaling forms $\rightarrow$ mean-field

Langevin eq. for order parameter $m(t)$

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = 3\lambda^2 m(t) - m(t)^3 + \eta(t) \ , \ \ \langle \eta(t)\eta(s)\rangle = 2T\delta(t-s)$$

contrôle parameter λ^2 :

(1) $\lambda^2 > 0 : T < T_c$, (2) $\lambda^2 = 0 : T = T_c$, (3) $\lambda^2 < 0 : T > T_c$

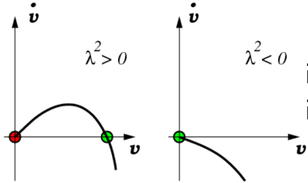
two-time observables : **response** $R(t,s)$, **correlation** $C(t,s)$

$$R(t,s) = \left.\frac{\delta\langle m(t)\rangle}{\delta h(s)}\right|_{h=0} = \frac{1}{2T}\langle m(t)\eta(s)\rangle \ , \ \ C(t,s) = \langle m(t)m(s)\rangle$$

mean-field equation of motion (cumulants neglected) :

$$\begin{aligned} \partial_t R(t,s) &= 3\left(\lambda^2 - v(t)\right) R(t,s) + \delta(t-s) \\ \partial_s C(t,s) &= 3\left(\lambda^2 - v(s)\right) C(t,s) + 2TR(t,s) \end{aligned}$$

with variance $v(t) = \langle m(t)^2\rangle$, $\dot{v}(t) = 6\big(\lambda^2 - v(t)\big)v(t)$



if $\lambda^2 \geq 0$: **fluctuations** persist
if $\lambda^2 < 0$: **fluctuations** disappear

in the long-time limit $t, s \to \infty$: $(t > s)$

$$R(t,s) \simeq \begin{cases} 1 \\ \sqrt{s/t} \\ e^{-3|\lambda^2|(t-s)} \end{cases} ; \; C(t,s) \simeq T \begin{cases} 2\min(t,s) & ; \; \lambda^2 > 0 \\ s\sqrt{s/t} & ; \; \lambda^2 = 0 \\ \frac{1}{(3|\lambda^2|)} e^{-3|\lambda^2|\,|t-s|} & ; \; \lambda^2 < 0 \end{cases}$$

fluctuation-dissipation ratio measures distance from equilibrium

$$X(t,s) = \frac{TR(t,s)}{\partial_s C(t,s)} \simeq \begin{cases} 1/2 + \mathrm{O}(e^{-6\lambda^2 s}) & ; \; \lambda^2 > 0 \\ 2/3 & ; \; \lambda^2 = 0 \\ 1 + \mathrm{O}(e^{-|\lambda^2|\,|t-s|}) & ; \; \lambda^2 < 0 \end{cases}$$

relaxation far from equilibrium, when $X \neq 1$, if $\lambda^2 \geq 0$ $(T \leq T_c)$

Consequences :

If $\lambda^2 > 0$: free random walk,
the system never reaches equilibrium !

If $\lambda^2 = 0$: slow relaxation, because of critical fluctuations

In both situations : observe

1. slow dynamics (non-exponential relaxation)
2. time-translation-invariance **broken**
3. **dynamical scaling behaviour**

$\longrightarrow$ the conditions for **physical ageing** are **all satisfied** if $T \leq T_c$

$\longrightarrow$ the system remains **out of equilibrium**

If $\lambda^2 < 0$: rapid relaxation, with finite relaxation time $\tau_{\rm rel} \sim 1/|\lambda^2|$, towards unique equilibrium state

Ageing exponents and X_∞ for quenches to $T = T_c$

model (model A dyn.)	d	$z(T_c)$	Θ	$\lambda_C(T_c)$	X_∞	
TDGL	1	2	0.199969	0.600616		E
Ising - KDH	1	4		1		E
Ising - Glauber	1	2	0	1	1/2	E
	2	2.1667(5)	0.191(3)	1.588(2)	0.328(1)	
	3	2.042(6)	0.108(2)	2.78(4)	0.4	
majority voter	2	2.170(5)	0.191(2)	1.595(10)		
Potts-3	2	2.197(3)	0.072(1)	1.85(4)	0.406(1)	
Potts-4	2	2.290(3)	−0.047(3)	2.27(5)	0.459(8)	
Turban-3	2	2.383(4)	−0.03(1)	2.32(5)	0.466(3)	
		2.292(4)	−0.047(8)	2.11		
Baxter-Wu	2	2.294(6)	−0.186(2)	2.6(1)	0.548(15)	
		1.994(24)	−0.185(2)	2.369(2)		
Turban-4	2	2.05(10)	−1.00(5)	—	—	$1^{\rm st}$
Blume-Capel	2	2.215(2)	−0.53(2)	3.17		
Ising FF	2	1.999(8)	0	2.006(10)	0.33(1)	
diluted Ising	3	2.62(7)	0.10(2)	2.75(7)	$\frac{1}{2} - \sqrt{\frac{3\varepsilon}{424}}$	
		2.2(2)	0.10(3)	2.73(30)		
clock-6	2	2.16(4)	0.254(5)	1.45		T_+
		2.24(2)	0.314(2)	1.29		T_-
XY	2	2 (log)	0.245(2)	1.494(5)	0.215(15)	
	3	≈ 2	0.16	2.68(10)	0.43(4)	
Heisenberg/dble exch.	3	1.976(9)	0.482(3)	2.04(3)		
spherical	<4	2	$1 - d/4$	$\frac{3}{2}d - 2$	$1 - 2/d$	E
	>4	2	0	d	1/2	E

in $2D$, the Potts-4, Turban-3 and Baxter-Wu models are in the same equilibrium universality class

in $1D$, the TDGL and the Ising-Glauber model are **distinct**

ageing exponents for quenches to $T < T_c$ (model A)

model	d	z	λ_C	a	class
Ising	2	2	1.246(20)	1/2	S
	3	2	1.60(2)	0.5	S
Potts-3	2	2	1.19(3)	0.49	S
Potts-8	2	2	1.25(1)	0.51	S
XY	3	2	1.7(1)	1/2	S
spherical	> 2	2	$d/2$	$d/2 - 1$	L
spherical, long-range	$> \sigma$	σ	$d/2$	$d/\sigma - 1$	L

for the O(n)-model :

$$\lambda_C = \frac{d}{2} + \left(\frac{4}{3}\right)^d (d+2)\frac{2d}{9} B\left(1+\frac{d}{2}, 1+\frac{d}{2}\right)\frac{1}{n} + \mathrm{O}(n^{-2})$$

MH & Pleimling, *Non-equilibrium phase transitions, vol. 2*, Springer (2010)

II. Local scaling with $z = 2 \rightarrow$ LSI

Question : extend dynamical scaling to larger set of dynamical symmetries, for given $z \neq 1$? MH 92, 94, 97, 02

motivation :

1. conformal invariance in equilibrium critical phenomena, $z = 1$
2. Schrödinger-invariance of simple diffusion, $z = 2$

(Jacobi 1842/43), Lie 1881, Appell 1892, Goff 27, Kastrup 68, Hagen 71, Niederer 72

$$t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta} \;,\;\; \mathbf{r} \mapsto \frac{\mathcal{R}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta} \;,\;\; \alpha\delta - \beta\gamma = 1$$

Lie algebra $\mathfrak{sch}(1) = \mathrm{Lie}(Sch(1)) = \langle X_{\pm 1,0}, Y_{\pm 1/2}, M_0 \rangle$ generators :

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - \frac{n+1}{2} t^n r \partial_r - \frac{n(n+1)}{4} \mathcal{M} t^{n-1} r^2 - \frac{1}{2}(n+1) x t^n \\ Y_m &= -t^{m+1/2}\partial_r - \left(m + \frac{1}{2}\right) \mathcal{M} t^{m-1/2} r \\ M_n &= -t^n \mathcal{M} \end{aligned}$$

also contains 'phase changes' in the wave function! (**projective**)

Explanation of these generators :

$$
\begin{aligned}
X_{-1} &= -\partial_t && \text{time translation} \\
X_0 &= -t\partial_t - \frac{1}{2} r\partial_r && \text{dilatation} \\
X_1 &= -t^2\partial_t - tr\partial_r && \text{`special Schrödinger'} \\
Y_{-1/2} &= -\partial_r && \text{space translation} \\
Y_{1/2} &= -t\partial_r && \text{Galilei transformation}
\end{aligned}
$$

$\mathfrak{sch}(d)$ **not** 'semi-simple' : can have **projective** representations
extra phase factors, give additional terms in the generators

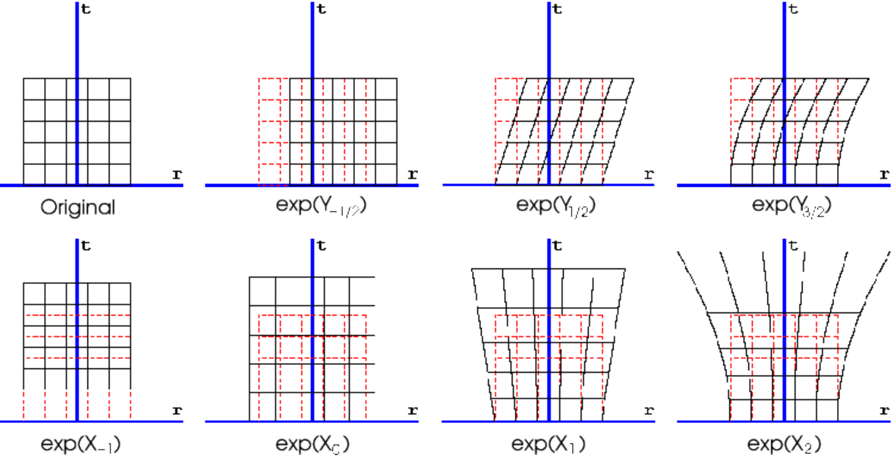
$$
\begin{aligned}
Y_{1/2} &= -t\partial_r - \mathcal{M} r \\
X_1 &= -t^2\partial_t - tr\partial_r - \frac{1}{2}\mathcal{M} r^2 \\
M_0 &= -\mathcal{M} && \text{phase shift}
\end{aligned}
$$

and also a **further generator** M_0 (central extension) :

$$[Y_{1/2}, Y_{-1/2}] = M_0$$

Finally, can have a scaling dimension x : extra terms in $X_{0,1}$.

Geometric illustration of a few Schrödinger transformations :



Hinrichsen '10

Lie algebra

non-vanishing commutators (including central extensions)

$$\begin{aligned}
[X_n, X_{n'}] &= (n-n')X_{n+n'} + \frac{c}{12}(n^3-n)\delta_{n+n',0} \\
[X_n, Y_m] &= \left(\frac{n}{2}-m\right) Y_{n+m} \\
[X_n, M_{n'}] &= -n' M_{n+n'} \\
[Y_m, Y_{m'}] &= (m-m')M_{m+m'}
\end{aligned}$$

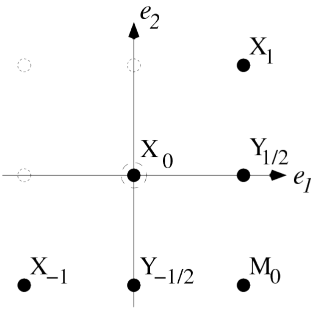
with $n, n' \in \mathbb{Z}$ and $m, m' \in \mathbb{Z} + \frac{1}{2}$

$\Rightarrow$ Schrödinger-Virasoro algebra $\mathfrak{sv}(1) \supset \mathfrak{vir}$

* contains 3 chiral fields, with $\dim X = 2$, $\dim Y = \frac{3}{2}$, $\dim M = 1$
* maximal finite-dimensional sub-algebra

$\Rightarrow$ Schrödinger algebra $\mathfrak{sch}(1) = \langle X_{\pm 1,0}, Y_{\pm 1/2}, M_0 \rangle \subset \mathfrak{sv}(1)$

mathematical results on deformations & central extensions — Roger & Unterberger '06 & '10 visualisation of commutators in a root diagramme (complexified)



$\mathfrak{sch}(1) = \langle X_{\pm 1,0}, Y_{\pm 1/2}, M_0 \rangle \subset B_2$

associate root vector $\mathbf{x} \longleftrightarrow X$ generator

vector addition $\mathbf{x} + \mathbf{x}' \longleftrightarrow [X, X']$ commutator

if $\mathbf{x} + \mathbf{x}' \not\in \text{diagramme}$, **then** $[X, X'] = 0$
if $\mathbf{x} + \mathbf{x}' = \mathbf{x}'' \in \text{diagramme}$, **then** $[X, X'] \sim X''$
(modulo generators from Cartan subalgebra $\mathfrak{h}$)

subalgebras $\longleftrightarrow$ convex set under vector addition
subalgebra **isomorphisms** $\longleftrightarrow$ discrete (Weyl) symmetries of diagramme

Dynamical symmetry I : $\mathfrak{sch}(d)$

1D Schrödinger operator : $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_r^2$

(free) Schrödinger/heat equation : $\boxed{\mathcal{S}\phi = 0}$

$$\begin{aligned} [\mathcal{S}, Y_{\pm 1/2}] &= [\mathcal{S}, M_0] = [\mathcal{S}, X_{-1}] = 0 \\ [\mathcal{S}, X_0] &= -\mathcal{S} \\ [\mathcal{S}, X_1] &= -2t\mathcal{S} + 2\mathcal{M}\left(x - \frac{1}{2}\right) \end{aligned}$$

infinitesimal change : $\delta\phi = \varepsilon\mathcal{X}\phi$, $\mathcal{X} \in \mathfrak{sch}(d), |\varepsilon| \ll 1$

Lemma : If $\mathcal{S}\phi = 0$ and $x = x_\phi = \frac{1}{2}$, then $\mathcal{S}(\mathcal{X}\phi) = 0$. NIEDERER '72

$\boxed{\mathfrak{sch}(d) \text{ maps solutions of } \mathcal{S}\phi = 0 \text{ onto solutions}}$.

Schrödinger-covariant two-point functions

two-point function $\boxed{R = R(t,s;\mathbf{r}_1,\mathbf{r}_2) := \langle \phi_1(t,\mathbf{r}_1)\widetilde{\phi}_2(s,\mathbf{r}_2)\rangle}$

physical assumption : co-variance under Schrödinger transformations
$\Rightarrow$ set of **linear** 1$^{\rm st}$-order differential eqs. : $\boxed{\mathcal{X}\mathbf{R} = \mathbf{0}}$; $\mathcal{X} \in \mathfrak{sch}(d)$
Each ϕ_i characterized by (i) scaling dimension x_i, (ii) mass $\mathcal{M}_i$

a) space & time translations : $R = R(\tau;\mathbf{r})$, $\tau = t - s$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$
b) Galilei $(1D)$:

$$\begin{aligned} Y_{1/2}R &= \left[-t_1\frac{\partial}{\partial r_1} - \mathcal{M}_1 r_1 - t_2\frac{\partial}{\partial r_2} - \mathcal{M}_2 r_2\right] R \\ &= \left[(-\tau\partial_r - \mathcal{M}_1 r) - r_2\left(\mathcal{M}_1 + \mathcal{M}_2\right)\right] R \stackrel{!}{=} 0 \end{aligned}$$

spatial translation-invariance $\Rightarrow$ any explicit reference to r_2 must disappear !

$$\begin{aligned} \left(-\tau\partial_r - \mathcal{M}_1 r\right) R(t,\mathbf{r}) &= 0 \qquad (1) \\ \left(\mathcal{M}_1 + \mathcal{M}_2\right) R(t,\mathbf{r}) &= 0 \qquad (2) \end{aligned}$$

$$R(\tau, \mathbf{r}) = f(\tau) \underbrace{\exp\left[-\frac{\mathcal{M}_1}{2}\frac{\mathbf{r}^2}{\tau}\right]}_{\text{heat kernel}} \underbrace{\delta(\mathcal{M}_1 + \mathcal{M}_2)}_{\textbf{Bargman rule}}$$

c) scaling : (use $\partial_i := \partial/\partial t_i$ and $D_i := \partial/\partial r_i$)

$$\begin{aligned} X_0 R &= \left[-t_1\partial_1 - \frac{1}{2}r_1 D_1 - t_2\partial_2 - \frac{1}{2}r_2 D_2 - \frac{1}{2}(x_1 + x_2)\right] R \\ &= \left[-\tau\partial_\tau - \frac{1}{2}r\partial_r - \frac{1}{2}(x_1 + x_2)\right] R \stackrel{!}{=} 0 \end{aligned}$$

hence $\boxed{f(\tau) = f_0 \tau^{-(x_1+x_2)/2}}$, $f_0 = \text{cste.}$

d) 'special' :

$$
\begin{aligned}
X_1 R &= \left[-t_1^2\partial_1 - t_2^2\partial_2 - t_1 r_1 D_1 - t_2 r_2 D_2 - \frac{\mathcal{M}_1}{2} r_1^2 - \frac{\mathcal{M}_2}{2} r_2^2 - x_1 t_1 - x_2 t_2\right] R \\
&= \Bigg[\left(-\tau^2\partial_\tau - \tau r\partial_r - \frac{\mathcal{M}_1}{2} r^2 - x_1\tau\right) - \frac{1}{2} r_2^2 \underbrace{(\mathcal{M}_1 + \mathcal{M}_2)}_{=0} \\
&\quad + 2t_2 \underbrace{\left(-\tau\partial_\tau - \frac{1}{2} r\partial_r - \frac{1}{2}(x_1 + x_2)\right)}_{=0} + r_2 \underbrace{(-\tau\partial_r - \mathcal{M}_1 r)}_{=0}\Bigg] R \\
&= \left[-\tau^2\partial_\tau - \tau r\partial_r - \frac{\mathcal{M}_1}{2} r^2 - x_1\tau\right] R(\tau, r) \stackrel{!}{=} 0
\end{aligned}
$$

use the decompositions $t_1^2 - t_2^2 = (t_1 - t_2)^2 + 2t_2(t_1 - t_2)$

$t_1 r_1 - t_2 r_2 = (t_1 - t_2)(r_1 - r_2) + t_2(r_1 - r_2) + r_2(t_1 - t_2)$

combine with previous conditions : $\boxed{\tau r(x_1 - x_2) R(\tau, r) = 0}$

$\boxed{f_0 = \delta_{x_1, x_2} r_0}$, with $r_0 = \text{cste.}$

MH '92, '94

Schrödinger-covariant three-point functions

two possible forms :

$$\langle \phi_1(t_1,\mathbf{r}_1)\phi_2(t_2,\mathbf{r}_2)\phi_3(t_3,\mathbf{r}_3)\rangle = \delta_{\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3,0}\ \exp\left[-\frac{\mathcal{M}_1}{2}\frac{\mathbf{r}_{13}^2}{t_{13}} - \frac{\mathcal{M}_2}{2}\frac{\mathbf{r}_{23}^2}{t_{23}}\right]$$
$$\times t_{13}^{-x_{13,2}/2} t_{23}^{-x_{23,1}/2} t_{12}^{-x_{12,3}/2}\, \Psi_{12,3}\left(\frac{(\mathbf{r}_{13}t_{23}-\mathbf{r}_{23}t_{13})^2}{t_{12}t_{13}t_{23}}\right)$$

$$\langle \phi_1(t_1,\mathbf{r}_1)\phi_2(t_2,\mathbf{r}_2)\phi_3(t_3,\mathbf{r}_3)\rangle = \delta_{\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3,0}\ \exp\left[-\frac{\mathcal{M}_2}{2}\frac{\mathbf{r}_{12}^2}{t_{12}} - \frac{\mathcal{M}_3}{2}\frac{\mathbf{r}_{13}^2}{t_{13}}\right]$$
$$\times t_{13}^{-x_{13,2}/2} t_{23}^{-x_{23,1}/2} t_{12}^{-x_{12,3}/2}\, \Psi_{1,23}\left(\frac{(\mathbf{r}_{13}t_{12}-\mathbf{r}_{12}t_{13})^2}{t_{12}t_{13}t_{23}}\right)$$

with $t_{ab} := t_a - t_b$, $\mathbf{r}_{ab} := \mathbf{r}_a - \mathbf{r}_b$ and $x_{ab,c} := x_a + x_b - x_c$
$\Psi_{12,3}$ and $\Psi_{1,23}$ are arbitrary differentiable functions

Ageing-covariant two-point functions I

$\mathfrak{sch}$-covariance **cannot** be used for ageing, since it contains time-translations X_{-1} !

restrict to **ageing algebra** $\boxed{\mathfrak{age}(1) := \left\langle X_{0,1}, Y_{\pm 1/2}, M_0 \right\rangle \subset \mathfrak{sch}(1)}$

NEW physical assumption : covariance under **ageing** transformations
$\Rightarrow$ set of **linear** $1^{\rm st}$-order differential eqs. : $\boxed{\mathcal{X}\mathbf{R} = \mathbf{0}}$; $\quad \mathcal{X} \in \mathfrak{age}(d)$
Each ϕ_i characterized by (i) scaling dimension x_i, (ii) mass $\mathcal{M}_i$

a) space translations : $R = R(u, v; \mathbf{r})$, $u = t - s$, $v = t/s$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

b) Galilei $(1D)$: $(-u\partial_r - \mathcal{M}_1 r)R - r_2(\mathcal{M}_1 + \mathcal{M}_2)R = 0$

$$R(u, v; \mathbf{r}) = f(u, v) \exp\left[-\frac{\mathcal{M}_1}{2}\frac{\mathbf{r}^2}{u}\right] \delta(\mathcal{M}_1 + \mathcal{M}_2)$$

c) scaling & special : (restrict to autoresponse, i.e. $\mathbf{r} = \mathbf{0}$)

$$\left(u\partial_u + \frac{1}{2}(x_1 + x_2) \right) \bar{R}(u,v) = 0$$

$$u \left(v\partial_v + \frac{x_1 - x_2}{2} \right) \bar{R}(u,v) = 0$$

the solution is found in a factorised form $\bar{R}(u,v) = r_1(u) r_2(v)$

$$\boxed{f(u,v) = r_0 u^{-(x_1+x_2)/2} v^{(x_2-x_1)/2}},$$

$r_0 = \text{cste.}$, **no** constraint on $x_{1,2}$

HPGL '01 ; MH '02

Ageing-covariant two-point functions II

$\mathfrak{age}(d)$ admits more general representations than $\mathfrak{sch}(d)$!

generalise form of X_n with $n \geq 0$: PICONE & MH 04 ; MH, ENSS, PLEIMLING 06

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n r\partial_r - \frac{n(n+1)}{4}\mathcal{M}t^{n-1}r^2 - \left[\frac{1}{2}(n+1)x + n\xi\right]t^n$$

physical assumption : covariance under **generalised** **ageing** transformations

$\Rightarrow$ set of **linear** $1^{\rm st}$-order differential eqs. : $\boxed{\mathcal{X}\mathbf{R} = \mathbf{0}}$; $\mathcal{X} \in \mathfrak{age}(d)$

Each ϕ_i characterized by (i) $1^{\rm st}$ scaling dimension x_i, (ii) $\mathbf{2}^{\rm nd}$ scaling dimension ξ_i, (iii) mass $\mathcal{M}_i$

a) space translations : $R = R(t, s; \mathbf{r})$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

b) Galilei : $R = r(t,s)\exp\left[-\frac{\mathcal{M}_1}{2}\frac{\mathbf{r}^2}{t-s}\right]\delta(\mathcal{M}_1 + \mathcal{M}_2)$

c) scaling & special : (with $y := t/s$)

$$r(t,s) = r_0 s^{-(x_1+x_2)/2} y^{\xi_2+(x_2-x_1)/2}(y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2}$$

Expected scaling form $R(t,s;\mathbf{r}) = s^{-1-a} f_R\left(\frac{t}{s}, \frac{\mathbf{r}}{(t-s)^{1/z}}\right)$
with $f_R(y,\mathbf{0}) \sim y^{-\lambda_R/z}$ for $y \gg 1$

$\mathfrak{age}(d)$-covariant two-point function : MH *et al.* '06

$$R(t,s;\mathbf{r}) = r_0 s^{-1-a} \left(\frac{t}{s}\right)^{1+a'-\lambda_R/z} \left(\frac{t}{s}-1\right)^{-1-a'} \exp\left(-\frac{\mathcal{M}_1}{2}\frac{\mathbf{r}^2}{t-s}\right)$$

with $1+a = \frac{x_1+x_2}{2}$, $a'-a = \xi_1+\xi_2$, $\lambda_R = 2(x_1+\xi_1)$, $\mathcal{M}_1+\mathcal{M}_2 = 0$

a) can derive **causality condition** $t > s$ MH & Unterberger '03
$\Rightarrow$ R is physically a **response function**

$$R(t,s;\mathbf{r}) = \lim_{h\to 0} \frac{\delta\langle\phi(t,\mathbf{r})\rangle}{\delta h(s,\mathbf{0})} = \left\langle \phi(t,\mathbf{r})\widetilde{\phi}(s,\mathbf{0})\right\rangle$$

b) in stochastic field-theory, the 'response field' $\widetilde{\phi}$ has formally a **negative** mass $\boxed{\mathcal{M}_{\widetilde{\phi}} = -\mathcal{M}_\phi}$ $\Longrightarrow$ Bargman rule explained

Dynamical symmetry II : $\mathfrak{age}(d)$

1D Schrödinger operator : $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_r^2 + 2\mathcal{M}\left(x + \xi - \frac{1}{2}\right) t^{-1}$

generalised 'Schrödinger equation' : $\boxed{\mathcal{S}\phi = 0}$

extra potential term arises in several models (e.g. spherical model)
if time-translations ($X_{-1} = -\partial_t$) are included, then $\xi = 0$

$$
\begin{aligned}
[\mathcal{S}, Y_{\pm 1/2}] &= [\mathcal{S}, M_0] = 0 \\
[\mathcal{S}, X_0] &= -\mathcal{S} \\
[\mathcal{S}, X_1] &= -2t\mathcal{S}
\end{aligned}
$$

infinitesimal change : $\delta\phi = \varepsilon\mathcal{X}\phi$, $\mathcal{X} \in \mathfrak{age}(d), |\varepsilon| \ll 1$

Lemma : If $\mathcal{S}\phi = 0$, then $\mathcal{S}(\mathcal{X}\phi) = 0$. Niederer '74 ; mh & Stoimenov '11

$\boxed{\mathfrak{age}(d) \text{ maps solutions of } \mathcal{S}\phi = 0 \text{ onto solutions}}$.

Dualisation

idée : treat the mass $\mathcal{M}$ as a variable, define 'dual' coordinate ζ

$$\phi(t,\mathbf{r}) = \phi_{\mathcal{M}}(t,\mathbf{r}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\zeta\, e^{-\mathrm{i}\mathcal{M}\zeta}\, \psi(\zeta, t, \mathbf{r})$$

trade projective representation for 'true' representation in auxiliary space

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - \frac{n+1}{2}\, t^n \mathbf{r}\cdot\partial_{\mathbf{r}} - (n+1)\frac{x}{2} t^n + \mathrm{i}\frac{n(n+1)}{4} t^{n-1}\mathbf{r}^2\partial_{\zeta} \\ \mathbf{Y}_m &= -t^{m+1/2}\partial_{\mathbf{r}} + \mathrm{i}\left(m+\frac{1}{2}\right) t^{m-1/2}\mathbf{r}\partial_{\zeta} \\ M_n &= \mathrm{i}t^n\partial_{\zeta} \end{aligned}$$

Generators live at the **boundary** of $(d+3)$-dim. Lorentzian space

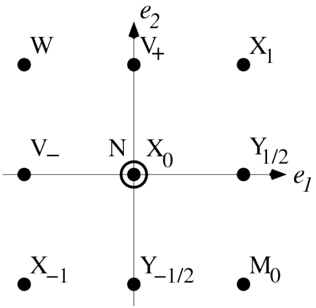
e.g. Minic & Pleimling 08, Fuertes & Moroz 09, Leigh & Hoang 09

The Schrödinger/heat equation becomes $\mathcal{S}\psi = 0$, explicitly

$$\mathcal{S}\psi = 2\mathrm{i}\frac{\partial^2\psi}{\partial\zeta\partial t} + \frac{\partial^2\psi}{\partial\mathbf{r}^2} = 0$$

visualisation of extension of $\mathfrak{sch}(1)$ from a root diagramme

$$\mathfrak{sch}(1) = \left\langle X_{\pm 1,0}, Y_{\pm 1/2}, M_0 \right\rangle \subset B_2 \cong \mathfrak{conf}(3)$$



new coordinates $\boldsymbol{\xi} = (\xi_{-1}, \xi_0, \xi_1)$

$$\zeta = \frac{1}{2}(\xi_0 + \mathrm{i}\xi_{-1}),\ t = \frac{1}{2}(-\xi_0 + \mathrm{i}\xi_{-1}),\ r = \sqrt{\frac{\mathrm{i}}{2}}\,\xi_1$$

Schrödinger/heat equation

$\partial_\mu \partial^\mu \Psi(\boldsymbol{\xi}) = 0$ with $\psi(\zeta, t, r) = \Psi(\boldsymbol{\xi})$

has conformal dynamical symmetry

$\Rightarrow$ include new generators $V_\pm, W, N$ — MH & Unterberger 03

in general, can extend $\mathfrak{sch}(d) \subset \mathfrak{conf}(d+2)_{\mathbb{C}}$

Burdet, Perrin, Sorba '73

explicit form of the new generators :

$$V_+ = -2tr\partial_t - 2\zeta r\partial_\zeta - (r^2 + 2\mathrm{i}\zeta t)\partial_r - 2xr \qquad V_- = -\zeta\partial_r + \mathrm{i}r\partial_t$$
$$W = -\zeta^2\partial_\zeta - \zeta r\partial_r + \frac{\mathrm{i}}{2}r^2\partial_t - x\zeta \qquad N = -t\partial_t + \zeta\partial_\zeta$$

$1D$ Schrödinger operator : $\mathcal{S} = 2M_0X_{-1} - Y^2_{-1/2} = 2\mathrm{i}\partial_\zeta\partial_t + \partial_r^2$

Schrödinger/heat equation $\boxed{\mathcal{S}\psi = 0}$

$$\begin{aligned} [\mathcal{S}, V_-] &= [\mathcal{S}, N] = 0 \\ [\mathcal{S}, V_+] &= 2(1-2x)\partial_t - 4r\mathcal{S} \\ [\mathcal{S}, W] &= \mathrm{i}(1-2x)\partial_r - 2\zeta\mathcal{S} \end{aligned}$$

infinitesimal change : $\delta\psi = \varepsilon\mathcal{X}\psi$, $\quad \mathcal{X} \in \mathfrak{conf}(3)_\mathbb{C}, |\varepsilon| \ll 1$

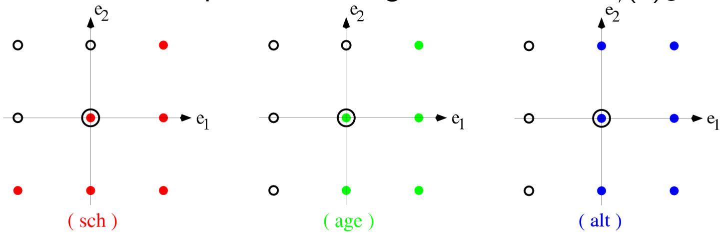
Lemma : If $\mathcal{S}\psi = 0$ and $x = x_\psi = \frac{1}{2}$, then $\mathcal{S}(\mathcal{X}\psi) = 0$.

$\boxed{\mathfrak{conf}(d+2)_\mathbb{C} \text{ maps solutions of } \mathcal{S}\psi = 0 \text{ onto solutions}}$

Parabolic subalgebras of B_2

Parabolic subalgebra : the sum of the Cartan subalgebra $\mathfrak{h}$ and the positive roots.
positive roots : all roots to the right of a straight line through $\mathfrak{h}$

Classification of parabolic subalgebras of $B_2 \cong \mathfrak{conf}(3)_{\mathbb{C}}$:



1. **extended ageing** $\widetilde{\mathfrak{age}}(1) := \mathfrak{age}(1) + \mathbb{C}N$
   $=$ **minimal** standard parabolic subalgebra
2. **extended Schrödinger** $\widetilde{\mathfrak{sch}}(1) := \mathfrak{sch}(1) + \mathbb{C}N$
3. **extended conformal Galilean (altern)** $\widetilde{\mathfrak{alt}}(1) := \mathfrak{alt}(1) + \mathbb{C}N$

MH & Unterberger, Nucl. Phys. **B660**, 407 (2003)

Physical consequence : causality

in auxiliary space, use conformal invariance $\langle \Psi_1(\boldsymbol{\xi}_1)\Psi_2(\boldsymbol{\xi}_2)\rangle = \Psi_0 \delta_{x_1,x_2} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^{-2x_1}$

$$\langle \psi_1(\zeta_1, t_1, \mathbf{r}_1)\psi_2(\zeta_2, t_2, \mathbf{r}_2)\rangle = \langle \Psi_1(\boldsymbol{\xi}_1)\Psi_2(\boldsymbol{\xi}_2)\rangle = \psi_0 \delta_{x_1,x_2} (t_1 - t_2)^{-x_1} \left(\zeta_1 - \zeta_2 + \frac{\mathrm{i}}{2}\frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{t_1 - t_2}\right)^{-x_1}$$

Physical convention : positive mass $\mathcal{M} > 0$ of field ϕ

If scaling dimension $x_1 > 0$, **then** derive causal form (2P) :

$$\langle \phi_1(t_1, \mathbf{r}_1)\phi_2^*(t_2, \mathbf{r}_2)\rangle = \int_{\mathbb{R}^2} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2 \, e^{-\mathrm{i}\mathcal{M}_1\zeta_1 + \mathrm{i}\mathcal{M}_2\zeta_2} \langle \psi_1(\zeta_1, t_1, \mathbf{r}_1)\psi_2(\zeta_2, t_2, \mathbf{r}_2)\rangle$$

$$= \quad \phi_0 \, \delta_{x_1,x_2} \, \delta_{\mathcal{M}_1,\mathcal{M}_2} \, \mathcal{M}_1^{1-x_1} \, \Theta(t_1 - t_2) \, (t_1 - t_2)^{-x_1} \exp\left(-\frac{\mathcal{M}_1}{2}\frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{t_1 - t_2}\right)$$

If scaling dimensions $x_1 > 0$, and $x_2 > 0$, **then** derive causal form (3P) :

$$\langle \phi_1(t_1, \mathbf{r}_1)\phi_2(t_2, \mathbf{r}_2)\phi_3^*(t_3, \mathbf{r}_3)\rangle = \mathcal{C}_{12,3} \, \delta(\mathcal{M}_1 + \mathcal{M}_2 - \mathcal{M}_3)$$
$$\times \quad \Theta(t_1 - t_3)\,\Theta(t_2 - t_3)\,(t_1 - t_2)^{-x_{12,3}/2}(t_1 - t_3)^{-x_{13,2}/2}(t_2 - t_3)^{-x_{23,1}/2}$$
$$\times \quad \exp\left[-\frac{\mathcal{M}_1}{2}\frac{(\mathbf{r}_1 - \mathbf{r}_3)^2}{t_1 - t_3} - \frac{\mathcal{M}_2}{2}\frac{(\mathbf{r}_2 - \mathbf{r}_3)^2}{t_2 - t_3}\right]$$
$$\times \quad \Psi_{12,3}\left(\frac{1}{2}\frac{[(\mathbf{r}_1 - \mathbf{r}_3)(t_2 - t_3) - (\mathbf{r}_2 - \mathbf{r}_3)(t_1 - t_3)]^2}{(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)}\right)$$

Causality requires at least the parabolic subalgebras of $\mathfrak{conf}(d + 2)_{\mathbb{C}}$

An infinite-dimensional extension of $\widetilde{\mathfrak{sch}}(1)$

extended Schrödinger-Virasoro algebra

$$\boxed{\widetilde{\mathfrak{sv}}(1) := \langle X_n, Y_m, M_n, N_n \rangle_{n\in\mathbb{Z}, m\in\mathbb{Z}+\frac{1}{2}} \supset \mathfrak{sv}(1)}$$

additional non-vanishing commutators, beyond those of $\mathfrak{sv}(1)$:

$$[X_n, N_{n'}] = -n' N_{n+n'} \ , \ \ [Y_m, N_n] = -Y_{m+n'} \ , \ \ [M_n, N_{n'}] = -2N_{n+n'}$$

admissible central extensions : $n, n' \in \mathbb{Z}$

$$\begin{aligned} [X_n, X_{n'}] &= (n-n')X_{n+n'} + \frac{c}{12}(n^3-n)\delta_{n+n',0} \\ [N_n, N_{n'}] &= \kappa n \delta_{n+n',0} \\ [X_n, N_{n'}] &= -n' N_{n+n'} + \alpha n^2 \delta_{n+n',0} \end{aligned}$$

maximal finite-dimensional sub-algebra : $\widetilde{\mathfrak{sch}}(1) = \mathfrak{sch}(1) + \mathbb{C}N_0$

C. Roger & J. Unterberger, Ann. H. Poincaré **7**, 1477 (2006) and Springer Lecture Notes (2011)

Stochastic field-theory

theoretical approach : **Langevin equation** (model A of Hohenberg & Halperin 77)

$$2\mathcal{M}\frac{\partial\phi}{\partial t} = \Delta\phi - \frac{\delta\mathcal{V}[\phi]}{\delta\phi} + \eta$$

order-parameter $\phi(t,\mathbf{r})$ non-conserved

$\mathcal{M}$: kinetic coéfficient $\mathcal{V}$: Landau-Ginsbourg potential

η : gaussian noise, centered and with variance

$$\langle\eta(t,\mathbf{r})\eta(t',\mathbf{r}')\rangle = 2T\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$$

fully disordered initial conditions (centred gaussian noise)

Langevin equations do **not** have non-trivial dynamical symmetries !

Galilei-invariance is broken by interactions with the thermal bath

dipole anisotropy of cosmic microwave background

? compare results of deterministic symmetries to stochastic models ?

take Langevin equation as classical equation of motion JANSSEN 92, DE DOMINICIS,...

$$\langle A\rangle = \int \mathcal{D}\phi\mathcal{D}\eta\, P[\eta]\,\delta\left((2\mathcal{M}\partial_t - \Delta)\phi + \mathcal{V}'[\phi] - \eta\right) A[\phi]$$

introduce auxiliary field $\widetilde{\phi}$, integrate out **gaussian** noise η

$\Rightarrow$ arrive at **effective field-theory**, with **action** $\mathcal{J}$ and averages

$$\langle A\rangle := \int \mathcal{D}\phi\mathcal{D}\widetilde{\phi}\, A[\phi,\widetilde{\phi}]\exp(-\mathcal{J}[\phi,\widetilde{\phi}])$$

$$\mathcal{J}[\phi,\widetilde{\phi}] = \underbrace{\int \widetilde{\phi}(2\mathcal{M}\partial_t - \Delta)\phi + \widetilde{\phi}\mathcal{V}'[\phi]}_{\mathcal{J}_0[\phi,\widetilde{\phi}]\ :\ \text{deterministic}} \underbrace{-T\int \widetilde{\phi}^2 - \int \widetilde{\phi}_{t=0}C_{init}\widetilde{\phi}_{t=0}}_{+\mathcal{J}_{\mathbf{b}}[\widetilde{\phi}]\ :\ \text{noise (}\mathbf{b}\text{ruit)}}$$

$\widetilde{\phi}$: response field ;

$$\boxed{C(t,s) = \langle\phi(t)\phi(s)\rangle,\ R(t,s) = \langle\phi(t)\widetilde{\phi}(s)\rangle}$$

deterministic averages : $\langle A\rangle_{\mathbf{0}} := \int \mathcal{D}\phi\mathcal{D}\widetilde{\phi}\, A[\phi,\widetilde{\phi}]\exp(-\mathcal{J}_{\mathbf{0}}[\phi,\widetilde{\phi}])$

masses :

$$\boxed{\mathcal{M}_\phi = -\mathcal{M}_{\widetilde{\phi}}}$$

Theorem : IF $\mathcal{J}_0$ *is Galilei- and spatially translation-invariant, then Bargman superselection rules* BARGMAN 54

$$\left\langle \phi_1 \cdots \phi_n \widetilde{\phi}_1 \cdots \widetilde{\phi}_m \right\rangle_{\mathbf{0}} \sim \delta_{n,m}$$

Illustration : computation of a response function PICONE & MH 04

$$\begin{aligned} R(t,s) &= \left\langle \phi(t)\widetilde{\phi}(s) \right\rangle = \left\langle \phi(t)\widetilde{\phi}(s) e^{-\mathcal{J}_b[\widetilde{\phi}]} \right\rangle_{\mathbf{0}} \\ &= \left\langle \phi(t)\widetilde{\phi}(s) \right\rangle_{\mathbf{0}} = R_{\mathbf{0}}(t,s) \end{aligned}$$

Bargman rule $\Longrightarrow$ response function does **not** depend on noise !
left side : computed in stochastic models
right side : local scale-symmetry of deterministic equation

Comparison of results of assumed deterministic $\mathfrak{age}(d)$-symmetry with explicit stochastic models/experiments **justified**.

choice of the (quasi-)primary operators ?

Finite transformation calculated from $\mathfrak{age}(d)$:

$t = \beta(t')$, $\mathbf{r} = \mathbf{r}'\sqrt{\frac{d\beta(t')}{dt'}}$ and $\beta(0) = 0$

$$\phi(t,\mathbf{r}) = \dot{\beta}(t')^{-x/2} \underbrace{\left(\frac{d\ln\beta(t')}{d\ln t'}\right)^{-\xi}}_{\text{extra transformation}} \underbrace{\exp\left[-\frac{\mathcal{M}\mathbf{r}'^2}{4}\frac{d\ln\dot{\beta}(t')}{dt'}\right]}_{\text{mass term}} \phi'(t',\mathbf{r}')$$

reduce to usual $\mathfrak{age}$-primary operator $\Phi(t,\mathbf{r}) := t^{-2\xi/z}\phi(t,\mathbf{r})$.

Then $\boxed{\Phi(t) = \dot{\beta}(t')^{-(x+2\xi)/z}\Phi'(t')}$, transforms as a $\mathfrak{sch}$-primary.

out of equilibrium, have **2** **distinct** scaling dimensions, $\boxed{x \text{ and } \xi}$.

Examples :

a) mean-field equation $\partial_t m = \Delta m + 3(\lambda^2 - v(t))m$ reduces to diffusion equation $\partial_t \Phi = \Delta \Phi$ via

$$m(t,\mathbf{r}) = \Phi(t,\mathbf{r}) \exp \int_0^t \mathrm{d}\tau\, 3\bigl(\lambda^2 - v(\tau)\bigr)$$

$$\text{two cases : } \begin{cases} \text{if } T = T_c \Leftrightarrow \lambda^2 = 0 : & \Phi(t) \sim t^{1/2} m(t) \\ \text{if } T < T_c \Leftrightarrow \lambda^2 > 0 : & \Phi(t) \sim 1 \cdot m(t) \end{cases}$$

$\Rightarrow$ **magnetisation** $m(t)$ **and** $\mathfrak{sch}$**-primary operator** $\Phi(t)$ **distinct**

b) kinetic spherical model equation, quenched to $T \leq T_c$

Godrèche & Luck '00

$$\partial_t \phi(t) = \Delta\phi(t) - v(t)\phi(t) + \text{noise} \;\;,\;\; v(t) \sim t^{-1}$$

gauge transformation $\Phi(t) = \phi(t) \exp\bigl[-\int_0^t \mathrm{d}\tau\, v(\tau)\bigr]$, gives diffusion eq. for Φ

c) kinetic Glauber-Ising model at $T = T_c$

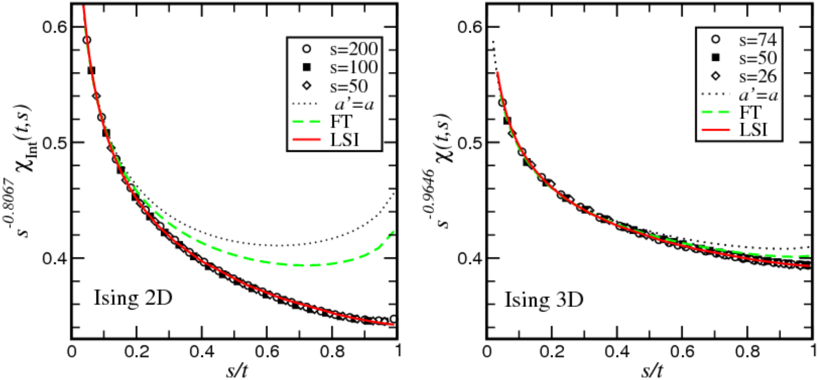
$1D$ **Glauber-Ising model**, at $T = 0$
exact two-time response function of the order-parameter
valid for both disordered and long-range initial conditions

LIPPIELLO & ZANNETTI 00, GODRÈCHE & LUCK 00, MH & SCHÜTZ 04

$$R(t,s;r) = R(t,s)\exp\left(-\frac{1}{4}\frac{r^2}{t-s}\right) \quad , \quad R(t,s) = \frac{1}{\pi}\sqrt{\frac{1}{2s(t-s)}}$$

read off : $a = 0, a' = -1/2, \lambda_R = 1, z = 2, \mathcal{M} = 1/2.$

Observation : the **hidden assumption** $a = a'$, uncritically taken over from equilibrium, is often **invalid** out of equilibrium.
Observables **cannot** always be identified with scaling operators.



LSI with $a \neq a'$: comparison with Ising data (momentum space !)
at $T = T_c$ and two-loop ε-expansion (FT) $\rightarrow$ resummation needed ?

One has $a' - a = -1/2$ in $1D$ (exact result) and
$a' - a = -0.187(20)$ in $2D$ and $a' - a = -0.022(5)$ in $3D$

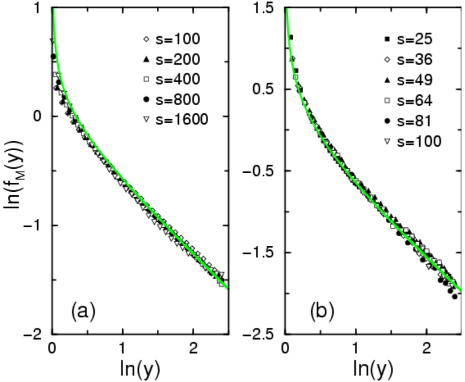
Pleimling & Gambassi, Phys. Rev. **B71**, 180401 ('05) ; mh, Enss, Pleimling, J. Phys. **A39**, L589 ('06)

Some known values of a, a' and λ_R/z at $T = T_c$.

model	d	a	$a' - a$	λ_R/z	Réf.
Ising	1	0	$-1/2$	$1/2$	Godrèche & Luck 00
	2	0.115	$-0.17(2)$	0.732(5)	H & P 03
	3	0.506	$-0.022(5)$	1.36(2)	H & P 03
EA spin glass	3	0.060(4)	$-0.76(3)$	0.38(2)	H & P 05
FA	1	1	$-3/2$	2	Mayer *et al* 06
	> 2	$1 + d/2$	-2	$2 + d/2$	Mayer *et al* 06
contact proc.	1	-0.681	0.270(10)	1.76(5)	H, E & P 06
NEKIM	1	$-0.430(2)$	0.00(1)	1.9(2)	Odor 06
voter Potts-3	2	≈ 0.11	-0.1	≈ 0.82	Chatelain *et al* 11
OJK model	≥ 2	$(d-1)/2$	$-1/2$	$d/4$	Mazenko 04

$\Longrightarrow$: $a \neq a'$ should be the generic case.

Tests of R in $2D/3D$ Glauber-Ising models



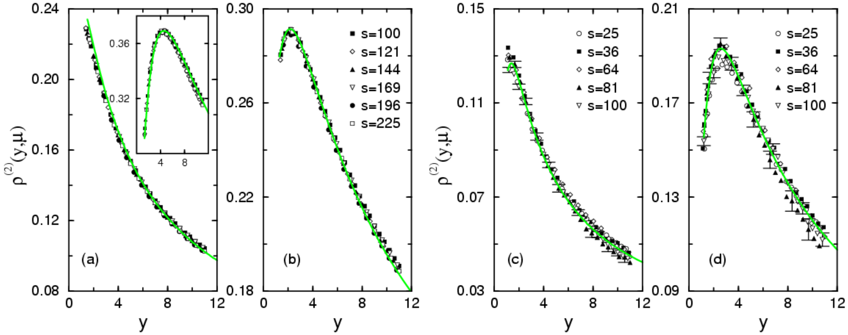
$$\chi_{\rm TRM}(t,s) = \int_0^s \mathrm{d}u\, R(t,u) = s^{-a} f_M(t/s)$$

integrated response (thermoremanent susceptibility) MH & PLEIMLING 03

$\chi_{\rm TRM}(t,s)$ for the Glauber-Ising model compared to LSI
(a) $2D$, $T=1.5$, (b) $3D$, $T=3$ $T<T_c$, hence $z=2$
compare data from **master equation** with local scale-symmetry

also **works** for (i) q-states $2D$ Potts model LORENZ & JANKE 07
(ii) $2D/3D$ XY model ABRIET & KAREVSKI 04

Test space-time behaviour **(parameter-free !)** :



spatio-temporally integrated response Ising model $T < T_c$

(a,b) $2D$; $\mu = 1, 2, 4$ (c,d) $3D$; $\mu = 1, 2,$

$$\int_0^s \mathrm{d}u \int_0^{\sqrt{\mu s}} \mathrm{d}r\, r^{d-1} R(t, u; \mathbf{r}) = s^{d/2-a} \rho^{(2)}(t/s, \mu)$$

MH & PLEIMLING, PHYS. REV. **E68**, 065101(R) (2003)

analogous results in the q-states $2D$ Potts model

LORENZ & JANKE, EUROPHYS. LETT. **77**, 10003 (2007)

III. Local scale-invariance for $z \neq 2$

Extend known cases $z = 1, 2 \Longrightarrow$ **axioms of LSI** :

MH 97/02, Baumann & MH 07

1. Möbius transformations in time (generator X_n)

$$t \mapsto t' = \frac{\alpha t + \beta}{\gamma t + \delta} \;\; ; \;\; \alpha\delta - \beta\gamma = 1$$

   require commutator : $[X_n, X_{n'}] = (n - n')X_{n+n'}$
2. Dilatation generator : $X_0 = -t\partial_t - \frac{1}{z}\mathbf{r}\cdot\partial_{\mathbf{r}} - \frac{x}{z}$
   Implies simple power-law scaling $L(t) \sim t^{1/z}$ (no glasses !).
3. Spatial translation-invariance $\rightarrow$ 2^e family Y_m of generators.
4. X_n contain phase terms from the scaling dimension $x = x_\phi$
5. X_n, Y_m contain further 'mass terms' (**Galilei** !)
6. finite number of independent conditions for n-point functions.

how to carry out this programme (outline) : decomposition

$$X_n = \underbrace{X_n^{(I)}}_{-t^{n+1}\partial_t} + \underbrace{X_n^{(II)}}_{-a_n(t,r)\partial_r} + \underbrace{X_n^{(III)}}_{-b_n(t,r)+\text{`mass'}} \quad , \quad Y_n = \underbrace{Y_n^{(I)}}_{=0} + Y_n^{(II)} + \underbrace{Y_n^{(III)}}_{\text{`mass'}}$$

Initial conditions : $a_{-1} = b_{-1} = 0$ and $a_0 = r/z$, $b_0 = x/z$.

Requirement : Consistency of the commutators ! **drop masses**

1. $[X_n, X_{-1}] \stackrel{!}{=} (n+1)X_{n-1}$ implies

$$\frac{\partial a_n}{\partial t} = (n+1)a_{n-1} \quad , \quad \frac{\partial b_n}{\partial t} = (n+1)b_{n-1}$$

2. $[X_n, X_0] \stackrel{!}{=} nX_n$ implies

$$\left(n + \frac{1}{z}\right) a_n = t\frac{\partial a_n}{\partial t} + \frac{r}{z}\frac{\partial a_n}{\partial r} \quad , \quad nb_n = t\frac{\partial b_n}{\partial t} + \frac{r}{z}\frac{\partial b_n}{\partial r}$$

3. $[X_n, X_1] \stackrel{!}{=} (n-1)X_{n+1}$ gives final form of a_n and b_n

$\Rightarrow$ solve these recurrences explicitly !

Theorem : LSI without 'masses'

Commutators $[X_n, X_{n'}] = (n - n')X_{n+n'}$, $[X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}$
with $n, n' \in \mathbb{Z}$ and $m \in \mathbb{Z} - 1/z$ have **only** the realisations :

z	$X_n = -t^{n+1}\partial_t - \frac{n+1}{z} t^n r\partial_r - \frac{(n+1)x}{z} t^n - \frac{n(n+1)}{2} B_{10} t^{n-1} r^z$ $Y_{k-1/z} = -t^k \partial_r - \frac{z^2}{2} k B_{10} t^{k-1} r^{-1+z}$
2	$X_n = -t^{n+1}\partial_t - \frac{1}{2}(n+1)t^n r\partial_r - \frac{1}{2}(n+1)xt^n - \frac{n(n+1)}{2} B_{10} t^{n-1} r^2 - \frac{(n^2-1)n}{6} B_{20} t^{n-2} r^4$ $Y_{k-1/2} = -t^k\partial_r - 2kB_{10}t^{k-1}r - \frac{4}{3}k(k-1)B_{20}t^{k-2}r^3$
1	$X_n = -t^{n+1}\partial_t - A_{10}^{-1}[(t+A_{10}r)^{n+1} - t^{n+1}]\partial_r - (n+1)xt^n - \frac{n+1}{2}\frac{B_{10}}{A_{10}}[(t+A_{10}r)^n - t^n]$ $Y_{k-1} = -(t+A_{10}r)^k\partial_r - \frac{k}{2}B_{10}(t+A_{10}r)^{k-1}$

free parameters (two in each case) : z, A_{10}, B_{10}, B_{20}

Three distinct algebras emerge :

1. generic z :

$$[X_n, X_{n'}] = (n-n')X_{n+n'} \quad , \quad [X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}$$

with $n \in \mathbb{Z}$ and $m \in \mathbb{Z} - 1/z$.

Only if $B_{10} = 0$: $\Longrightarrow [Y_m, Y_{m'}] = 0$.
In this case, **if** $z = 2/N$ and furthermore $N \in \mathbb{N}$, **then**
finite-dimensional subalgebra $\left\langle X_{\pm 1,0}, Y_{-N/2,-N/2+1,\ldots,N/2} \right\rangle$

MH, PHYS. REV. LETT. **78**, 1940 (1997)

called nowadays by string theorists 'spin-l algebra'

But if $B_{10} \neq 0$: **difficult** closure problem, see below.

2. $z=2$. The Schrödinger algebra. or $N=1$

then have **two** dimensionful parameters B_{10} and B_{20}

Find closed infinite-dimensional extension of $\mathfrak{sch}(1)$: MH '02

define **three** families of charges

$Z_n^{(0)} := -2t^n$, $Z_m^{(1)} := -2t^{m-1/2}r$ and $Z_n^{(2)} := -nt^{n-1}r^2$

$$
\begin{aligned}
[Y_m, Y_{m'}] &= (m-m')\left(4B_{20}Z^{(2)}_{m+m'} + B_{10}Z^{(0)}_{m+m'}\right) \\
[X_n, Z^{(0,2)}_{n'}] &= -n'Z^{(0,2)}_{n+n'} \ , \ [X_n, Z^{(1)}_m] = -(n/2-m)Z^{(1)}_{n+n'} \\
[Y_m, Z^{(1)}_{m'}] &= -Z^{(0)}_{m+m'} \ , \ [Y_m, Z^{(2)}_n] = -nZ^{(1)}_{m+n}
\end{aligned}
$$

Recover Schrödinger-Virasoro algebra

$\mathfrak{sv}(1) = \langle X_n, Y_m, M_n\rangle_{n\in\mathbb{Z}, m\in\mathbb{Z}+\frac{1}{2}} \supset \mathfrak{sch}(1)$ for $B_{20}=0$ and $B_{10}=\mathcal{M}/2$.

? physical applications of these infinite-dimensional symmetries **?**

3. $z=1$. Around the conformal galilean algebra or $N=2$. MH '07,'02

Then $[Y_n, Y_{n'}] = A_{10}(n-n')Y_{n+n'}$, in $d=1$ dimensions.

* If $A_{10}\neq 0$, then **isomorphic** to $\mathfrak{vect}(S^1)\times\mathfrak{vect}(S^1)\cong\mathfrak{conf}(2)$.

$$X_n = \ell_n + \bar{\ell}_n \quad , \quad Y_n = A_{10}\bar{\ell}_n$$

* Invariant Schrödinger operator $\mathcal{S} = -A_{10}\partial_t + \partial_r$. (with $x = B_{10}/2A_{10}$)
* Set $A_{10} =: \mu$ and $B_{10} =: 2\gamma$.

Quasi-primary operator ϕ_i characterised by the triplett (x_i, μ_i, γ_i).

$$\langle\phi_1\phi_2\rangle = \delta_{x_1,x_2}\delta_{\mu_1,\mu_2}\delta_{\gamma_1,\gamma_2}\; f_0\, t_{12}^{-2x_1}\left(1+\mu_1\frac{r_{12}}{t_{12}}\right)^{-2\gamma_1/\mu_1}$$

$$\langle\phi_1\phi_2\phi_3\rangle = f_{123}\, t_{13}^{-x_{13,2}}\, t_{23}^{-x_{23,1}}\, t_{12}^{-x_{12,3}}\left(1+\mu\frac{r_{13}}{t_{13}}\right)^{-\gamma_{13,2}/\mu}\left(1+\mu\frac{r_{23}}{t_{23}}\right)^{-\gamma_{23,1}/\mu}\left(1+\mu\frac{r_{12}}{t_{12}}\right)^{-\gamma_{12,3}/\mu}$$

and **Bargman rule** $\mu_1=\mu_2=\mu_3=:\mu$ **universal constant**.

Distinct from the two- and three-point functions of conformal invariance.

In the limit $A_{10} = \mu \to 0$, **contraction** to **altern-Viraosoro algebra**

$$\boxed{\mathfrak{av}(1) \supset \mathfrak{alt}(1) \equiv \mathrm{CGA}(1)}.$$

or 'full **conformal galilean algebra**' HAVAS & PLEBANSKI '78, MH '97, NEGRO *et al.* '97,...

In d space dimensions, generators of $\mathfrak{av}(d) \supset \mathrm{CGA}(d) \equiv \mathfrak{alt}(d)$ ($\gamma \in \mathbb{R}^d$)

$$\begin{aligned}
X_n &= -t^{n+1}\partial_t - (n+1)t^n \mathbf{r}\cdot\nabla - (n+1)t^n x - n(n+1)t^{n-1}\boldsymbol{\gamma}\cdot\mathbf{r} \\
Y_n^{(j)} &= -t^{n+1}\partial_j - (n+1)t^n\gamma_j \\
R_0^{(jk)} &= -\left(r_j\partial_k - r_k\partial_j\right) - \left(\gamma_j\partial_{\gamma_k} - \gamma_k\partial_{\gamma_j}\right); \qquad j \neq k
\end{aligned}$$

CHERNIHA & MH '10

with abbreviations $\partial_j = \frac{\partial}{\partial r_j}$. Non-vanishing commutators :

$$[X_n, X_m] = (n-m)X_{n+m}\ ,\ \ [X_n, Y_m^{(j)}] = (n-m)Y_{n+m}^{(j)}\ ,\ \ [R_0^{(jk)}, Y_m^{(\ell)}] = \delta^{j,\ell}\, Y_m^{(k)} - \delta^{k,\ell}\, Y_m^{(j)}$$

* two Virasoro-like **independent** central charges OVSIENKO & ROGER 98

* **contract** two- & three-point functions (limit $\mu \to 0$), find

MH '02 ; MARTELLI & TASHIKAWA 09, BAGCHI, MANDAL & GOPAKUMAR 09, HOSSEINY & ROUHANI 10,...

$$\langle \phi_1 \phi_2 \rangle = \delta_{x_1,x_2} \delta_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2} \, f_0 \, t_{12}^{-2x_1} \exp\left[-2 \frac{\gamma_1 \cdot \mathbf{r}_{12}}{t_{12}} \right]$$

$$\langle \phi_1 \phi_2 \phi_3 \rangle = f_{123} \, t_{13}^{-x_{13,2}} t_{23}^{-x_{23,1}} t_{12}^{-x_{12,3}} \exp\left[-\frac{\boldsymbol{\gamma}_{12,3} \cdot \mathbf{r}_{12}}{t_{12}} - \frac{\boldsymbol{\gamma}_{23,1} \cdot \mathbf{r}_{23}}{t_{23}} - \frac{\boldsymbol{\gamma}_{31,2} \cdot \mathbf{r}_{31}}{t_{31}} \right]$$

with $x_{ij,k} := x_i + x_j - x_k$ and $\gamma_{ij,k} := \gamma_i + \gamma_j - \gamma_k$.

* For $d = 2$ so-called **exotic** central extension of $\mathrm{CGA}(2)$, but incompatible with ∞-dim. extension of $\mathrm{CGA}(2) \subset \mathfrak{av}(2)$

LUKIERSKI, STICHEL, ZAKREWSKI 06/07

known (conditionally) invariant non-linear hydrodynamic equations ($\neq$ Navier-Stokes) ZHANG & HÓRVATHY '09, CHERNIHA & MH '10

* *similar classification* from a geometric point of view, using the Newton-Cartan formalism DUVAL & HÓRVATHY 09

A possible construction of mass terms for generic z (set $B_{10} = 0$)

Extend to $z \neq 1, 2$ by generators with mass terms, for $d = 1$:

$$Y_{1-1/z} := -t\partial_r - \mu z r \nabla_{\mathbf{r}}^{2-z} - \gamma z(2-z)\partial_r \nabla_{\mathbf{r}}^{-z} \qquad \text{Galilei}$$

$$X_1 := -t^2\partial_t - \frac{2}{z} tr\partial_r - \frac{2(x+\xi)}{z} t - \mu r^2 \nabla_{\mathbf{r}}^{2-z} - 2\gamma(2-z) r\partial_r \nabla_{\mathbf{r}}^{-z} - \gamma(2-z)(1-z)\nabla_{\mathbf{r}}^{-z} \qquad \text{special}$$

- depend on two parameters γ, μ and on two dimensions x, ξ
- contains fractional derivative ($\widehat{f}$: Fourier transform)

$$\nabla_{\mathbf{r}}^{\alpha} f(\mathbf{r}) := \mathrm{i}^{\alpha} \int_{\mathbb{R}^d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{\alpha} e^{\mathrm{i}\mathbf{r}\cdot\mathbf{k}} \widehat{f}(\mathbf{k})$$

- some properties : $\nabla_{\mathbf{r}}^{\alpha}\nabla_{\mathbf{r}}^{\beta} = \nabla_{\mathbf{r}}^{\alpha+\beta}$, $[\nabla_{\mathbf{r}}^{\alpha}, r_i] = \alpha\partial_{r_i}\nabla_{\mathbf{r}}^{\alpha-2}$
  $\nabla_{\mathbf{r}}^{\alpha} \exp(\mathrm{i}\mathbf{q}\cdot\mathbf{r}) = \mathrm{i}^{\alpha}|\mathbf{q}|^{\alpha} \exp(\mathrm{i}\mathbf{q}\cdot\mathbf{r})$

Fact 1 : simple algebraic structure :

$$[X_n, X_{n'}] = (n-n')X_{n+n'} \quad , \quad [X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}$$

$\rightarrow$ Generate Y_m from $Y_{-1/z} = -\partial_r$.

Fact 2 : LSI-invariant Schrödinger operator :

$$\mathcal{S} := -\mu\partial_t + z^{-2}\nabla_{\mathbf{r}}^{z}$$

Let $x_0 + \xi = 1 - 2/z + (2-z)\gamma/\mu$. Then $[\mathcal{S}, Y_m] = 0$ and

$$[\mathcal{S}, X_0] = -\mathcal{S} \quad , \quad [\mathcal{S}, X_1] = -2t\mathcal{S} + \frac{2\mu}{z}(x - x_0)$$

$\Longrightarrow$ $\boxed{\mathcal{S}\phi = 0}$ is **lsi-invariant** equation, if $x_\phi = x_0$.

Physical assumption (hidden & approximate) : equations of motion remain of first order in ∂_t, even after renormalisation.

Fact 3 : non-trivial conservation laws :

iterated commutator with $G := Y_{1-1/z}$, $\mathrm{ad}\,_G. = [., G]$

$$M_\ell := (\mathrm{ad}_G)^{2\ell+1}\, Y_{-1/z} = a_\ell \mu^{2\ell+1} \nabla_{\mathbf{r}}^{(2\ell+1)(1-z)+1}$$

For $z = 2$, $a_\ell = 0$ if $\ell \geq 1$. For a n-point function $F^{(n)} = \langle \phi_1 \ldots \phi_n \rangle$, $M_\ell F^{(n)} = 0$ gives in momentum space

$$\left(\sum_{i=1}^{n} \mu_i^{2\ell-1} |\mathbf{k}_i|^{2\ell-(2\ell-1)z} \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) = 0$$

$$\left(\sum_{i=1}^{n} \mathbf{k}_i \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) = 0$$

$\Longrightarrow$ momentum conservation & conservation of $|\mathbf{k}|^\alpha$!

analogous to relativistic factorisable scattering Zamolodchikov[2] 79, 89

equil. analogy : $2D$ Ising model at $T = T_c$ in magnetic field

<u>**Consequence :**</u> a $\mathfrak{lsi}$-covariant $2n$-point function $F^{(2n)}$ is only non-zero, if the 'masses' μ_i can be arranged in pairs $(\mu_i, \mu_{\sigma(i)})$ with $i = 1, \ldots, n$ such that $\boxed{\mu_i = -\mu_{\sigma(i)}}$.

generalised Galilei-invariance with $z \neq 2 \Longrightarrow$ integrability

Corollary 1 : Bargman rule : $\langle \phi_1 \ldots \phi_n \widetilde{\phi}_1 \ldots \widetilde{\phi}_m \rangle_0 \sim \delta_{n,m}$

Corollary 2 : treat (linear) stochastic equations with $\mathfrak{lsi}$-invariant deterministic part, reduction formulæ

Corollary 3 : response function noise-independent

$$
\begin{aligned}
R(t,s;\mathbf{r}) &= R(t,s)\mathcal{F}^{(\mu_1,\gamma_1)}(|\mathbf{r}|(t-s)^{-1/z}) \\
R(t,s) &= r_0\, s^{-a} \left(\frac{t}{s}\right)^{1+a'-\lambda_R/z} \left(\frac{t}{s}-1\right)^{-1-a'} \\
\mathcal{F}^{(\mu,\gamma)}(\mathbf{u}) &= \int_{\mathbb{R}^d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^\gamma \exp\left(\mathrm{i}\mathbf{u}\cdot\mathbf{k} - \mu|\mathbf{k}|^z\right)
\end{aligned}
$$

Corollary 4 :

Correlators obtained from factorised 4-point responses.

How to test the foundations of LSI

theory is built on :

a) simple scaling – domain sizes $L(t) \sim t^{1/z}$

b) invariance under Möbius transformation $t \mapsto t/(\gamma t + \delta)$

c) Galilei-invariance generalised to $z \neq 2$

together with spatial translation-invariance

$\Longrightarrow$ extended Bargman rules

$\Longrightarrow$ factorisation of $2n$-point functions

Möbius transformation	autoresponse $R(t, s)$
generalised Galilei-invariance	space-time response $R(t, s; \mathbf{r})$
factorisation	two-time correlation function

Correlation functions for $z = 2$

find $C(t,s) = \langle \phi(t)\phi(s)\rangle = \langle \phi(t)\phi(s) e^{-\mathcal{J}_b[\widetilde{\phi}]}\rangle_0$ from Bargman rule

$$
\begin{aligned}
C(t,s) &= \frac{a_0}{2}\int_{\mathbb{R}^d} \mathrm{d}\mathbf{R}\, R_0^{(3)}(t,s,0;\mathbf{R}) && \text{initial} \\
&\quad + \frac{T}{2\mathcal{M}}\int_0^\infty \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}\mathbf{R}\, R_0^{(3)}(t,s,u;\mathbf{R}) && \text{thermal} \\
R_0^{(3)}(t,s,u;\mathbf{r}) &= \left\langle \phi(t,\mathbf{y})\phi(s,\mathbf{y})\widetilde{\phi}^2(u,\mathbf{r}+\mathbf{y})\right\rangle_0
\end{aligned}
$$

$\mathfrak{sch}(d)$-invariance fixes three-point $R_0^{(3)}$ function up to an unknown scaling function $\Psi \Longrightarrow$ how to obtain a prediction for $f_C(y)$?

Theorem : LSI with $z = 2 \Longrightarrow \boxed{\lambda_C = \lambda_R}$ PICONE & MH 04

agrees with a different argument of BRAY and with all models

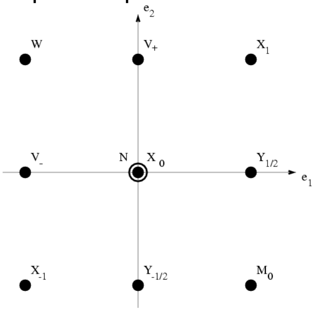
hypotheses : a) consider $\mathcal{M}$ as a further variable GIULINI 96

b) extend $\mathfrak{sch}(d)$ to conformal algebra $\mathfrak{conf}(d+2)$

BURDET ET AL 73, MH & UNTERBERGER 03

$1D$ Schrödinger equation $\to$ $3D$ Laplace equation

new generators $N, V_\pm, W$



MH, PICONE, PLEIMLING 04

conformal inv. of diffusion equation :
$[\mathcal{S}, N] = [\mathcal{S}, V_-] = 0$

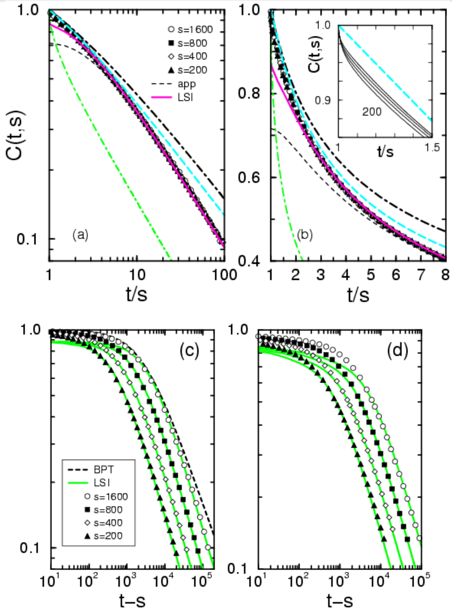
extra conditions : $N R_0^{(3)} = V_- R_0^{(3)} = 0$
fix Ψ.

$\Longrightarrow f_C(y)$ explicitly known

simple special case : free field-theory : PICONE & MH 04, MH & BAUMANN 07

$$f_C(y) \approx \begin{cases} \left[(y+1)^2/(4y)\right]^{-\lambda_C/2} & ; \quad T < T_c \\ \int_0^1 \mathrm{d}v\, v^{\lambda_C - 2 - 2a' - 2\mu} \left[(y-v)(1-v)\right]^{a'-b-2\mu} & ; \quad T = T_c \\ \times (y+1-2v)^{b-2a'-1+2\mu}\, y^{1+a'-\lambda_C/2} & \end{cases}$$

NB : $\widetilde{\phi}^2$ treated a composite field $\Longrightarrow \mu$ free parameter



Autocorrelation in the $2D$ Ising model, $T < T_c$

LSI : prediction from $\mathfrak{conf}(3)$

BPT : gaussian closed form

Bray & Puri 91, Toyoki 92

LM : perturbative schemes

Liu & Mazenko 91, Mazenko 98

app : free-field approximation

lower row :

left : $T = 0$, **right :** $T = 1.5$

mh, Picone, Pleimling, Europhys. Lett. **68**, 191 (2004)

also works for q-states $2D$ Potts model

Lorenz & Janke 07

Test in the $1D$ **Glauber-Ising model**, at $T = T_c = 0$:

$$C(t,s) = \frac{2}{\pi} \arctan \sqrt{\frac{2}{t/s - 1}} \qquad \text{exact} \quad \text{L\& Z 00, G\& L 00, H\& S 04}$$

$$\stackrel{!}{=} C_0 \int_0^1 \mathrm{d}v\, v^{2\mu} \left[\left[\frac{t}{s} - 1 \right] (1-v) \right]^{-2\mu - 1/2} \left[\frac{t}{s} + 1 - 2v \right]^{2\mu}$$

choose $\mu = -1/4$ and $C_0 = \sqrt{2}/\pi$.

similarly : (i) spherical model, (ii) XY model for $T \to 0$ (spin waves) (iii) linear voter model (iv) random walk

Conclusion :

- **no** local scaling in **full** Langevin equation
- local scaling in **deterministic** part $\to$ reduction formulæ
- **hidden** local scaling symmetry, at least when $z = 2$
- physical origin of Galilei-invariance ?

Correlators obtained from factorised 4-point responses :

$$C(t,s) = \langle \phi(t)\phi(s)\rangle = \langle \phi(t)\phi(s) e^{-\mathcal{J}_b[\widetilde{\phi}]}\rangle_{\mathbf{0}}$$

example : contribution of 'initial' noise at time u :

$$\begin{aligned} C_{\rm init}(t,s;\mathbf{r}) &= \int_{\mathbb{R}^{2d}} \mathrm{d}\mathbf{R}\mathrm{d}\mathbf{R}' \underbrace{F^{(4)}(t,s,u;\mathbf{r},\mathbf{R},\mathbf{R}')}_{\text{4-pt function}} \underbrace{\mathbf{C}(u,\mathbf{R}-\mathbf{R}')}_{\text{'initial' correlator}} \\ &= c_0\,(ts)^{2\xi/z+F} s^{4\widetilde{x}/z-2F}(t-s)^{-2(2\xi+x)/z} \\ &\quad \times \int_{\mathbb{R}^d} \mathrm{d}\mathbf{k}\, |\mathbf{k}|^{2\beta} \exp\left[\mathrm{i}\mathbf{r}\cdot\mathbf{k} - \alpha|\mathbf{k}|^z(t-s)\right] \widehat{\mathbf{C}}(s,\mathbf{k}) \end{aligned}$$

where we have also sent $u \to s$.
Relevant, e.g. for **phase-ordering kinetics** $\to$ $z=2$ Bray & Rutenberg 94

Ising model, more precise 'initial' correlator : Ohta, Jasnow, Kawasaki '82

$$\mathbf{C}(t;\mathbf{r}) = \frac{2}{\pi}\arcsin\left(\exp\left[-\frac{\mathbf{r}^2}{L(t)^2}\right]\right)$$

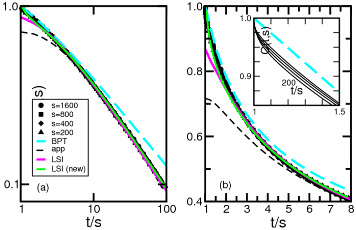
2D Ising model, $T < T_c$: autocorrelator in the scaling limit

$$C(ys,s) = C_0 y^{\rho}(y-1)^{-\rho-\lambda_C/z} \int_0^{\infty} \mathrm{d}x\, e^{-x} f_{\nu}\left(\sqrt{\frac{x}{y-1}}\right)$$

$$f_{\nu}(\sqrt{u}) = \int_0^{\infty} \mathrm{d}v \,\arcsin\left(e^{-\nu v}\right) J_0(\sqrt{uv}\,)$$

parameters to be fitted : ρ, ν.

$z = 2$



of practical importance :
'good' choice of 'initial' correlations
$C_{\rm ini}(\mathbf{r}) = c_0\delta(\mathbf{r})$ not sufficient

BAUMANN & MH 10

$\Longrightarrow$ for the first time, a theoretical calculation for $C(t,s)$ reproduces the simulations for **all** t/s !

IV. Recent extensions

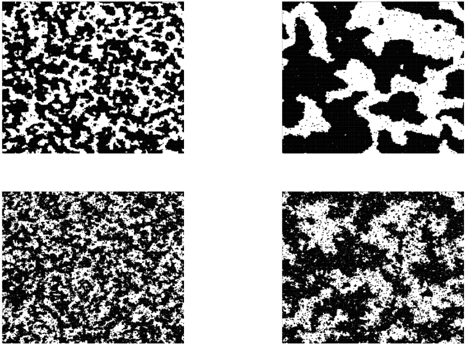
A) logarithmic extension of $\mathfrak{age}$-invariance

arxiv.1009.4139

B) non-local representations

mh & S. Stoimenov, Nucl. Phys. **B847**, 612 (2011)

$t = t_1$ $t = t_2 > t_1$

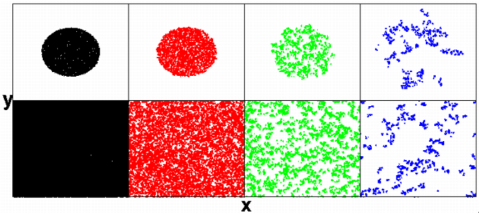


magnet $T < T_c$

$\longrightarrow$ordered cluster

magnet $T = T_c$

$\longrightarrow$correlated cluster



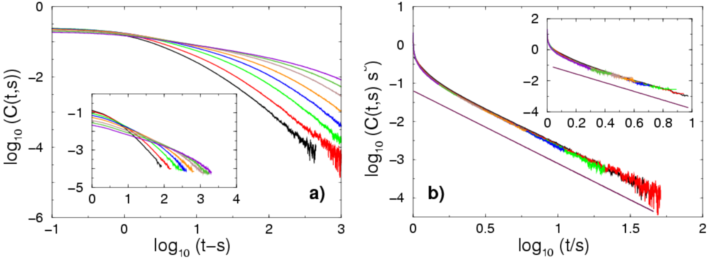
critical contact process

$\Longrightarrow$**cluster dilution**

voter model, contact process,. . .

A.1 Critical contact process (directed percolation)

ageing and scaling for $C(t,s)$: **critical** contact process



main figures : $1D$, insets : $2D$

observe all **3** properties of **ageing** : $\left\{\begin{array}{l}\text{slow dynamics}\\ \text{no TTI}\\ \text{dynamical scaling}\end{array}\right.$

<u>contrast</u> to critical magnets : $\boxed{a \neq b}$ $\Longrightarrow$ **no** finite FDR !

Ramasco, mh, Santos, da Silva Santos 04 ; Enss, mh, Schollwöck 04

numerical values of some non-equilibrium exponents

contact process (CP) $A \to 2A, A \to \emptyset$, parity-conserved model (PC) $A \leftrightarrow 3A, 2A \to \emptyset$, diffusion-coagulation (DC) $2A \to A$

	d	a	b	λ_C/z	λ_R/z		
CP	1	$-0.68(5)$	0.32(5)	1.85(10)	1.85(10)	TMRG	[1]
		$-0.57(10)$	0.3189	1.9(1)	1.9(1)	MC	[2]
		-0.6810			1.76(5)	MC	[3]
		-0.6810	0.3189	1.7921	1.7921	scal	[5]
	2	0.3(1)	0.901(2)	2.8(3)	2.75(10)	MC	[2]
		$-0.198(2)$	0.901(2)	2.58(2)	2.58(2)	scal	[5]
			0.9(1)	2.5(1)		exp	[6]
	>4	$d/2-1$	$d/2$		$d/2+2$	MF	[2]
PC	1	$-0.430(4)$	0.570(4)	1.9(1)	1.9(2)	MC	[4]
		$-0.430(4)$	0.570(4)	1.86(1)	1.86(1)	scal	
DC	1	$-1/2$	1	2	2	exact	[7]

[1] Enss *et. al.* 04 ; [2] Ramasco *et. al.* 04 ; [3] Hinrichsen 06 ; [4] Ódor 06 ;

[5] Baumann & Gambassi 07 ; [6] Takeuchi *et. al.* 09 ; [7] Durang, Fortin, mh 11

in the contact process $\boxed{1+a=b}$: $\Longleftarrow$ **rapidity-reversal symmetry** of stationary state of CP $\Rightarrow$ **specific property** !

why does $1+a=b$ also hold in the PC class **?**

$\Longrightarrow$ try **new form of FDR** ! ENSS *et. al.* 04 ; BAUMANN & GAMBASSI 07

$$\Xi(t,s) := \frac{R(t,s)}{C(t,s)} = \frac{f_R(t/s)}{f_C(t/s)} \ , \quad \Xi_\infty := \lim_{s\to\infty}\left(\lim_{t\to\infty}\Xi(t,s)\right)$$

universal function, $\frac{1}{\Xi} \neq 0$ measures distance to stationary state

in $d = 4 - \varepsilon$ dimensions, from an one-loop calculation B & G 07

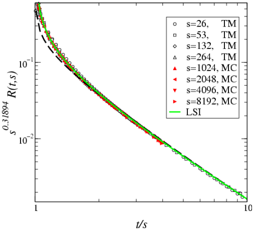
$$\Xi_\infty = 2\left[1 - \varepsilon\left(\frac{119}{480} - \frac{\pi^2}{120}\right)\right] + \mathrm{O}(\varepsilon^2)$$

quantitatively consistent with TMRG estimate $\Xi_\infty = 1.15(5)$ in $1D$.

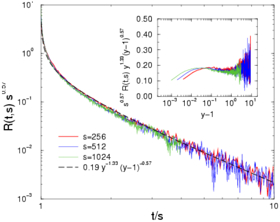
NB : $1+a=b$ **invalid** in other non-equilibrium universality classes $\Rightarrow$ need different forms of FDR ! BAUMANN *et. al.* 05 ; DURANG & MH 09, DURANG *et al.* 11

Particle models : comparison of $R(t,s)$ with LSI-prediction :

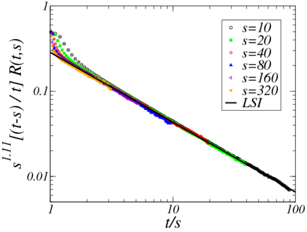
contact process (CP)



nonequil. kinetic Ising (PC)



voter Potts-3 (VP3)



CP : $a' - a \simeq 0.27$

PC : $a' - a \simeq 0.00(1)$

VP3 : $a' - a \simeq -0.1$

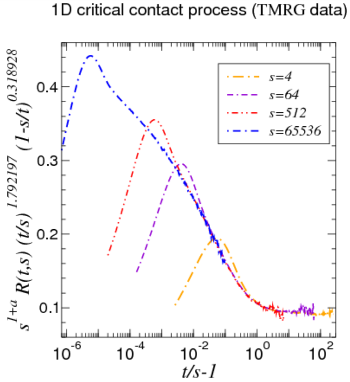
MH, ENSS, PLEIMLING 06
ENSS 06 ; HINRICHSEN 06

ÓDOR 06

CHATELAIN *et al.* 11

? is this good general agreement already conclusive ?

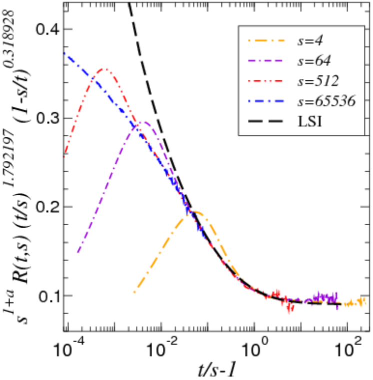
Observation : the **hidden assumption** $a = a'$, uncritically taken over from equilibrium, is often **invalid** out of equilibrium.
Observables **cannot** always be identified with scaling operators.



study more closely the limit $t, s \to \infty$, $y = t/s$ fixed ; let $y \to 1$

$$R(t,s) = s^{-1-a} f_R\left(\frac{t}{s}\right) \;,\; h_R(y) := f_R(y) y^{\lambda_R/z} (1 - 1/y)^{1+a}$$

observe good collapse of data, when $y = t/s$ large enough
LSI with $a = a'$ predicts : $h_R(y) = f_0 = \text{cste.}$
$\Rightarrow$ reproduces TMRG data for $y \gtrsim 3 - 4$



$$h_R(y) := f_R(y) y^{\lambda_R/z} (1 - 1/y)^{1+a} \stackrel{\text{LSI}}{=} f_0 (1 - 1/y)^{a-a'}$$

with the choice $a' - a = 0.26$, LSI works well for $y \gtrsim 1.1$
but **systematic deviations**, still **inside the ageing scaling region**, for smaller values of $y = t/s$ (down to $y \simeq 1.001$) !

Question : improve the prediction of local scale-invariance (LSI) ?

A.2 Logarithmic conformal invariance

generalise conformal invariance → doubletts $\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$

scalars : **generators** : $\ell_n = -w^{n+1}\partial_w - (n+1)w^n\Delta$,

Δ : **conformal weight**

commutator : $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$; $n, m \in \mathbb{Z}$

invariance : Laplace equation $\boxed{\mathcal{S}\psi = \partial_w\partial_{\bar{w}}\psi = 0}$

is conformally invariant for $\Delta = \overline{\Delta} = 0$ since

$$[\mathcal{S}, \ell_n] = -(n+1)w^n\mathcal{S} - (n+1)nw^{n-1}\Delta\partial_{\bar{w}}$$

doubletts : Gurarie '93

generators $\quad \ell_n = -w^{n+1}\partial_w - (n+1)w^n\begin{pmatrix} \Delta & 1 \\ 0 & \Delta \end{pmatrix}$

'Laplace' equation $\quad \mathcal{S}\Psi = \begin{pmatrix} 0 & \partial_w\partial_{\bar{w}} \\ 0 & 0 \end{pmatrix}\Psi = 0$

invariance $\quad [\mathcal{S}, \ell_n] = -(n+1)w^n\mathcal{S} - (n+1)nw^{n-1}\begin{pmatrix} 0 & \Delta \\ 0 & 0 \end{pmatrix}\partial_{\bar{w}}$

define **two-point correlators :** GURARIE '93, RAHIMI TABAR *et al.* '97

$$F := \langle \phi_1(w_1)\phi_2(w_2)\rangle \ , \ G := \langle \phi_1(w_1)\psi_2(w_2)\rangle \ , \ H := \langle \psi_1(w_1)\psi_2(w_2)\rangle$$

(a) translation-invariance (ℓ_{-1}) :
$F = F(w), G = G(w), H = H(w),$ $w = w_1 - w_2$
(b) dilatation-invariance & special invariance for $F(w)$

$$\left.\begin{array}{ll} \ell_0 : & (-w\partial_w - \Delta_1 - \Delta_2)\, F(w) = 0 \\ \ell_1 : & \left(-w^2\partial_w - 2w\Delta_1\right) F(w) = 0 \end{array}\right\} \Rightarrow w(\Delta_1 - \Delta_2)F(w) = 0$$

if $F(w) \neq 0$, then $\boxed{\Delta_1 = \Delta_2}$.
(c) dilatation-invariance & special invariance for $G(w)$

$$\left.\begin{array}{ll} \ell_0 : & (-w\partial_w - \Delta_1 - \Delta_2)\, G(w) = F(w) \\ \ell_1 : & \left(-w^2\partial_w - 2w\Delta_1\right) G(w) = 0 \end{array}\right\} \Rightarrow (\Delta_1 - \Delta_2)G(w) = F(w)$$

one has : $\boxed{F(w) = 0 \textbf{ and } \Delta_1 = \Delta_2}$.

(d) dilatation-invariance & special invariance for $H(w)$

with $\Delta := \Delta_1 = \Delta_2$

$$\left. \begin{array}{ll} \ell_0: & (-w\partial_w - 2\Delta)\, H(w) = G(w) + G(-w) \\ \ell_1: & \left(-w^2\partial_w - 2w\Delta\right) H(w) = 2wG(w) \end{array} \right\} \Rightarrow G(w) = G(-w)$$

Consequences :

$$G(w) = G(-w) = G_0|w|^{-2\Delta}$$

$$w\frac{\mathrm{d}H(w)}{\mathrm{d}w} + 2\Delta H(w) + 2G_0|w|^{-2\Delta} = 0$$

and finally

$$H(w) = (H_0 - 2G_0 \ln|w|)\,|w|^{-2\Delta}$$

Logarithmic conformal invariance has been found in

- critical $2D$ percolation — CARDY '92, WATTS 96, MATHIEU & RIDOUX '07-'08
- disordered systems — CAUX *et al.* '96
- sand-pile models — RUELLE *et al.* '08-'10

A.3 Logarithmic Schrödinger-invariance

as for logarithmic conformal invariance, construct doubletts $\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$

Formally, scaling dimension x becomes a Jordan matrix :

$$x \mapsto \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$$

can repeat exactly the same calculation to find co-variant two-point (reponse) functions :

$$\begin{aligned} F &:= \langle \phi_1(t_1, \mathbf{r}) \phi_2(t_2, \mathbf{0}) \rangle \ , \ G := \langle \phi_1(t_1, \mathbf{r}) \psi_2(t_2, \mathbf{0}) \rangle \ , \\ H &:= \langle \psi_1(t_1, \mathbf{r}) \psi_2(t_2, \mathbf{0}) \rangle \end{aligned}$$

and one obtains, with $t = t_1 - t_2$ Hosseiny & Rouhani '10

$$\begin{aligned} F &= 0 \ , \ G = G_0 |t|^{-x} \exp\left[-\frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t} \right] \ , \\ H &= \left(H_0 - G_0 \ln|t| \right) |t|^{-x} \exp\left[-\frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t} \right] \end{aligned}$$

A.4 Logarithmic ageing-invariance

Schrödinger-invariance cannot be a dynamical symmetry for ageing, since it contains time-translations X_{-1} !

Go to **ageing algebra** $\mathfrak{age}(d) := \left\langle X_{1,0}, Y^{(j)}_{\pm 1/2}, M_0, R^{(jk)}_0 \right\rangle_{j,k=1,\ldots d}$
Need generalised form of generator

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2} t^n \mathbf{r}\cdot\boldsymbol{\nabla}_{\mathbf{r}} - \frac{\mathcal{M}}{2}(n+1)nt^{n-1}\mathbf{r}^2 - \frac{n+1}{2} x t^n - (n+1)n\xi t^n$$

construct **logarithmic ageing-invariance** by the formal changes :

$$x \mapsto \left(\begin{array}{cc} x & x' \\ 0 & x \end{array} \right) , \; \xi \mapsto \left(\begin{array}{cc} \xi & \xi' \\ \xi'' & \xi \end{array} \right)$$

concentrate on time-dependence

$$X_0 = -t\partial_t - \frac{1}{2}\begin{pmatrix} x & x' \\ 0 & x \end{pmatrix} \quad , \quad X_1 = -t^2\partial_t - t\begin{pmatrix} x+\xi & x'+\xi' \\ \xi'' & x+\xi \end{pmatrix}$$

and compute commutator

$$[X_1, X_0] = X_1 + \tfrac{1}{2}t\,x'\xi''\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \stackrel{!}{=} X_1 \implies \boxed{x'\xi'' \stackrel{!}{=} 0}$$

$x' = 0$: either, $\begin{pmatrix} \xi & \xi' \\ \xi'' & \xi \end{pmatrix} \to \begin{pmatrix} \xi_+ & 0 \\ 0 & \xi_- \end{pmatrix}$ is diagonalisable
$\Rightarrow$ non-logarithmic case.
Or else, it reduces to a Jordan form $\Rightarrow$ 2^{nd} case.

$\xi'' = 0$: simultaneous Jordan forms $\Rightarrow$ generic case.
(one can arrange for $x' = 0$ or $x' = 1$).

$$\boxed{\text{we can always arrange for } \xi'' = 0.}$$

invariant Schrödinger equation $\mathcal{S}\Psi = 0$, with :

$$\mathcal{S} := \left(2\mathcal{M}\partial_t - \boldsymbol{\nabla}_{\mathbf{r}}^2 + \frac{2\mathcal{M}}{t}\left(x + \xi - \frac{d}{2} \right) \right) \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)$$

If $x + \xi = d/2$, have also log-invariance under $\mathfrak{sch}(d)$.

Co-variant two-point functions :

$$\begin{aligned} F = F(t_1, t_2) &:= \langle \phi_1(t_1)\phi_2(t_2) \rangle \\ G_{12} = G_{12}(t_1, t_2) &:= \langle \phi_1(t_1)\psi_2(t_2) \rangle \\ G_{21} = G_{21}(t_1, t_2) &:= \langle \psi_1(t_1)\phi_2(t_2) \rangle \\ H = H(t_1, t_2) &:= \langle \psi_1(t_1)\psi_2(t_2) \rangle \end{aligned}$$

co-variance conditions (with $\partial_i = \partial/\partial t_i$) :

$$
\begin{aligned}
\left[t_1\partial_1 + t_2\partial_2 + \frac{1}{2}(x_1+x_2)\right] F(t_1,t_2) &= 0 \\
\left[t_1^2\partial_1 + t_2^2\partial_2 + (x_1+\xi_1)t_1 + (x_2+\xi_2)t_2\right] F(t_1,t_2) &= 0 \\
\left[t_1\partial_1 + t_2\partial_2 + \frac{1}{2}(x_1+x_2)\right] G_{12}(t_1,t_2) + \frac{x_2'}{2} F(t_1,t_2) &= 0 \\
\left[t_1^2\partial_1 + t_2^2\partial_2 + (x_1+\xi_1)t_1 + (x_2+\xi_2)t_2\right] G_{12}(t_1,t_2) + (x_2'+\xi_2')t_2 F(t_1,t_2) &= 0 \\
\left[t_1\partial_1 + t_2\partial_2 + \frac{1}{2}(x_1+x_2)\right] G_{21}(t_1,t_2) + \frac{x_1'}{2} F(t_1,t_2) &= 0 \\
\left[t_1^2\partial_1 + t_2^2\partial_2 + (x_1+\xi_1)t_1 + (x_2+\xi_2)t_2\right] G_{21}(t_1,t_2) + (x_1'+\xi_1')t_1 F(t_1,t_2) &= 0 \\
\left[t_1\partial_1 + t_2\partial_2 + \frac{1}{2}(x_1+x_2)\right] H(t_1,t_2) + \frac{x_1'}{2} G_{12}(t_1,t_2) + \frac{x_2'}{2} G_{21}(t_1,t_2) &= 0 \\
\left[t_1^2\partial_1 + t_2^2\partial_2 + (x_1+\xi_1)t_1 + (x_2+\xi_2)t_2\right] H(t_1,t_2) & \\
+(x_1'+\xi_1')t_1 G_{12}(t_1,t_2) + (x_2'+\xi_2')t_2 G_{21}(t_1,t_2) &= 0
\end{aligned}
$$

8 eqs. for 4 functions in 2 variables $\Rightarrow$ expect **unique solution**, up to normalisations.

Solve these via the following **ansatz**, with $\boxed{y := t_1/t_2}$.

Set $\mathcal{F}(y) := y^{\xi_2+(x_2-x_1)/2}(y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2}$. Then

$$\begin{aligned} F(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\,\mathcal{F}(y)\,f(y) \\ G_{12}(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\,\mathcal{F}(y)\sum_{j\in\mathbb{Z}} \ln^j t_2 \cdot g_{12,j}(y) \\ G_{21}(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\,\mathcal{F}(y)\sum_{j\in\mathbb{Z}} \ln^j t_2 \cdot g_{21,j}(y) \\ H(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\,\mathcal{F}(y)\sum_{j\in\mathbb{Z}} \ln^j t_2 \cdot h_j(y) \end{aligned}$$

must find the functions $f,\ g_{12,j},\ g_{21,j},\ h_j$; where $j \in \mathbb{Z}$

Results :

(1) : $f(y) = f_0 = \mathrm{cste}.$ standard form of LSI

(2) : consider G_{12}. Dilatation-covariance (X_0) gives

$$\left(g_{12,1}(y) + \frac{1}{2}x_2' f(y)\right) + \sum_{j\neq 0}(j+1)\ln^j t_2 \cdot g_{12,j+1}(y) = 0$$

Must hold true for all times t_2. The only non-vanishing terms are :

$$g_{12}(y) := g_{12,0}(y) \ , \ \ \gamma_{12}(y) := g_{12,1}(y) = -\frac{1}{2}x_2' f(y)$$

Co-variance under the special transformations (X_1) gives

$$\sum_{j\in\mathbb{Z}} \ln^j t_2 \left(y(y-1)\frac{\mathrm{d}g_{12,j}(y)}{\mathrm{d}y} + (j+1)g_{12,j+1}(y)\right) + (x_2' + \xi_2')f(y) = 0$$

for all times t_2 and leads to

$$y(y-1)\frac{\mathrm{d}g_{12}(y)}{\mathrm{d}y} + \left(\frac{x_2'}{2} + \xi_2'\right) f(y) = 0$$

(3) : consider G_{21}. We find the only non-vanishing terms

$$g_{21}(y) := g_{21,0}(y) \ , \ \ \gamma_{21}(y) := g_{21,1}(y) = -\frac{1}{2}x_1' f(y)$$

and the differential equation

$$y(y-1)\frac{\mathrm{d}g_{21}(y)}{\mathrm{d}y} + \left(x_1' + \xi_1'\right) yf(y) - \frac{1}{2}x_1' f(y) = 0$$

(4) : consider H. We find the only non-vanishing terms $h_0(y)$ and

$$\begin{aligned} h_1(y) &= -\frac{1}{2}\left(x_1' g_{12}(y) + x_2' g_{21}(y)\right) \\ h_2(y) &= \frac{1}{4}x_1' x_2' f(y) \end{aligned}$$

and the differential equation

$$y(y-1)\frac{\mathrm{d}h_0(y)}{\mathrm{d}y} + \left(\left(x_1' + \xi_1'\right) y - \frac{1}{2}x_1'\right) g_{12}(y) + \left(\frac{1}{2}x_2' + \xi_2'\right) g_{21}(y) = 0$$

The remaining differential equations have the solutions :

$$
\begin{aligned}
g_{12}(y) &= g_{12,0} + \left(\frac{x_2'}{2} + \xi_2'\right) f_0 \ln\left|\frac{y}{y-1}\right| \\
g_{21}(y) &= g_{21,0} - \left(\frac{x_1'}{2} + \xi_1'\right) f_0 \ln|y-1| - \frac{x_1'}{2} f_0 \ln|y| \\
h_0(y) &= h_0 - \left[\left(\frac{x_1'}{2} + \xi_1'\right) g_{21,0} + \left(\frac{x_2'}{2} + \xi_2'\right) g_{12,0}\right] \ln|y-1| - \left[\frac{x_1'}{2} g_{21,0} - \left(\frac{x_2'}{2} + \xi_2'\right) g_{12,0}\right] \ln|y| \\
&\quad + \frac{1}{2} f_0 \left[\left(\left(\frac{x_1'}{2} + \xi_1'\right) \ln|y-1| + \frac{x_1'}{2} \ln|y|\right)^2 - \left(\frac{x_2'}{2} + \xi_2'\right)^2 \ln^2\left|\frac{y}{y-1}\right|\right]
\end{aligned}
$$

where $f_0, g_{12,0}, g_{21,0}, h_0$ are normalisation constants. **Summary :**

$$
\begin{aligned}
F(t_1, t_2) &= t_2^{-(x_1+x_2)/2}\, \mathcal{F}(y)\, f_0 \\
G_{12}(t_1, t_2) &= t_2^{-(x_1+x_2)/2}\, \mathcal{F}(y) \left(g_{12}(y) - \ln t_2 \cdot \frac{x_2'}{2} f_0 \right) \\
G_{21}(t_1, t_2) &= t_2^{-(x_1+x_2)/2}\, \mathcal{F}(y) \left(g_{21}(y) - \ln t_2 \cdot \frac{x_1'}{2} f_0 \right) \\
H(t_1, t_2) &= t_2^{-(x_1+x_2)/2}\, \mathcal{F}(y) \Big(h_0(y) - \ln t_2 \cdot \frac{1}{2}(x_1' g_{12}(y) + x_2' g_{21}(y)) \\
&\qquad + \ln^2 t_2 \cdot \frac{x_1' x_2'}{4} f_0 \Big)
\end{aligned}
$$

Retour to ageing phenomena

we find the co-variant two-point (auto-response) functions (with $y = t/s$) :

$$\begin{aligned}
\left\langle \phi(t)\widetilde{\phi}(s) \right\rangle &= s^{-(x+\widetilde{x})/2} f(y) \\
\left\langle \phi(t)\widetilde{\psi}(s) \right\rangle &= s^{-(x+\widetilde{x})/2} \left(g_{12}(y) + \ln s \cdot \gamma_{12}(y)\right) \\
\left\langle \psi(t)\widetilde{\phi}(s) \right\rangle &= s^{-(x+\widetilde{x})/2} \left(g_{21}(y) + \ln s \cdot \gamma_{21}(y)\right) \\
\left\langle \psi(t)\widetilde{\psi}(s) \right\rangle &= s^{-(x+\widetilde{x})/2} \left(h_0(y) + \ln s \cdot h_1(y) + \ln^2 s \cdot h_2(y)\right)
\end{aligned}$$

all scaling functions explicitly known

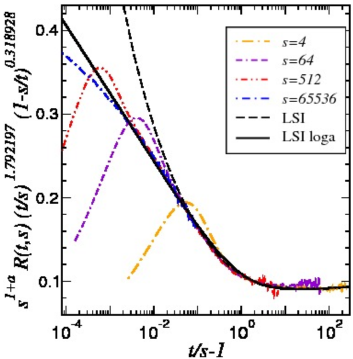
Question : $1D$ **directed percolation described by logarithmic** LSI **?**

as motivated by the applications of logarithmic conformal invariance to $2D$ critical normal percolation — Mathieu & Ridoux '07-08

assumption : $R(t,s) = \left\langle \psi(t)\widetilde{\psi}(s) \right\rangle$

good collapse $\Rightarrow$ **no** logarithmic corrections $\Rightarrow$ $\boxed{x' = \widetilde{x}' = 0}$

$$h_R(y) = \left(1 - \frac{1}{y}\right)^{a-a'} \left[h_0 - g_{12,0}\widetilde{\xi}' \ln(1-1/y) - g_{21,0}\xi' \ln(y-1) - \frac{1}{2} f_0 \widetilde{\xi}'^2 \ln^2(1-1/y) + \frac{1}{2} f_0 \xi'^2 \ln^2(y-1) \right]$$



find empirically :
very small amplitude of
$\ln^2$-terms

$\Rightarrow$ $\boxed{f_0 = 0}$

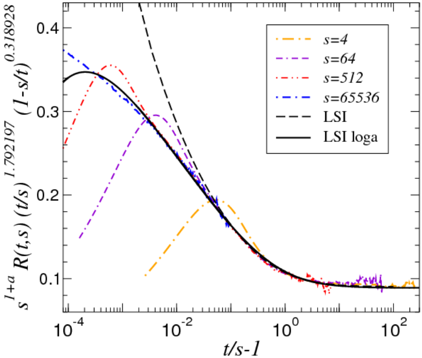
require both $\xi \neq 0$, $\widetilde{\xi}' \neq 0$

logar. LSI works at least down to $y \simeq 1.002$, with $a' - a \simeq -0.002$.

An alternative interpretation : $R(t,s) = \left\langle \psi(t)\widetilde{\psi}(s) \right\rangle$

good collapse $\Rightarrow$ **no** logarithmic corrections $\Rightarrow$ $\boxed{x' = \widetilde{x}' = 0}$

$$h_R(y) = \left(1 - \frac{1}{y}\right)^{a-a'} \left[h_0 - g_{12,0}\widetilde{\xi}' \ln(1-1/y) - \frac{1}{2} f_0 \widetilde{\xi}'^2 \ln^2(1-1/y) - g_{21,0}\xi' \ln(y-1) + \frac{1}{2} f_0 \xi'^2 \ln^2(y-1) \right]$$



no logarithmic growth for $y \to \infty$

$\Rightarrow$ $\boxed{\xi' = 0}$

only $\widetilde{\xi}' \neq 0$ remains !

logar. LSI works at least down to $y \simeq 1.005$, with $a' - a \simeq 0.17$.

B.1. Non-local representations of $\mathfrak{age}(1)$, $z = n \neq 2$

existing local scale-transformations for $z \neq 2$ have in general generators of higher than first order

Consider simple situation with $z = n \neq 2$ and $d = 1$: $(n \in \mathbb{N})$

$$
\begin{aligned}
X_0 &= -\frac{n}{2} t\partial_t - \frac{1}{2} r\partial_r - \frac{x}{2} \\
X_1 &= -\frac{n}{2} t^2 \partial_t \partial_r^{n-2} - tr\partial_r^{n-1} - \frac{1}{2}\mu r^2 - (x+\xi) t\partial_r^{n-2} \\
Y_{-1/2} &= -\partial_r \\
Y_{1/2} &= -t\partial_r^{n-1} - \mu r \\
M_0 &= -\mu
\end{aligned}
$$

Schrödinger operator :

$$
\mathcal{S} = n\mu \frac{\partial}{\partial t} - \frac{\partial^n}{\partial r^n} + 2\mu \left(x + \xi + \frac{n-1}{2} \right) \frac{1}{t}
$$

Dynamical symmetry :

$$[\mathcal{S}, X_0] = -\frac{n}{2}\mathcal{S} \ , \ \ [\mathcal{S}, X_1] = -nt\partial_r^{n-2}\mathcal{S}$$

Commutator relations of $\mathfrak{age}(1)$ are satisfied, with one exception :

$$\left[X_1, Y_{1/2}\right] = \frac{n-2}{2} t^2 \partial_r^{n-3}\mathcal{S}$$

Function space : construct **equivalence classes** with respect to Schrödinger equation $\boxed{\mathcal{S}\phi = 0}$

For each $n \in \mathbb{N}$, have on the **restricted space** of solutions of $\mathcal{S}\phi = 0$ a representation of $\mathfrak{age}(1)$. Contains standard space-translations and dilatations (vector fields), but Galilei- and special transformations are **non-local**.

formal Lie series : $F(\epsilon, t, r) = e^{-\epsilon Y_{1/2}} F(0, t, r)$, similarly for X_1.

Gives the initial-value problems :

$$\left(\partial_\epsilon - t\partial_r^{n-1} - \mu r\right) F(\epsilon, t, r) = 0 \ , \ \text{for } Y_{1/2}$$

$$\left(\partial_\epsilon - \frac{n}{2} t^2 \partial_t \partial_r^{n-2} - tr\partial_r^{n-1} - xt\partial_r^{n-2} - \frac{1}{2}\mu r^2\right) F(\epsilon, t, r) = 0 \ , \ \text{for } X_1$$

with the initial condition $F(0, t, r) = \phi(t, r)$. In particular :

time coordinate : $\phi(t, r) = t$ with $x = \xi = 0$, $\mu = 0$

space coordinate : $\phi(t, r) = r$ with $x = \xi = 0$, $\mu = 0$

by analogy with the standard representation with $z = n = 2$.

Rigid transformations & standard Galilei-transformations

$$(\partial_\epsilon - t\partial_r - \mu r)\, F(\epsilon, t, r) = 0 \ \ , \ \ F(0, t, r) = \phi(t, r) \ \ , \ \ n = 2$$

In Fourier space :

$$\widehat{\phi}(t, k) \mapsto \widehat{F}(\epsilon, t, k) = \widehat{\phi}(t, k + \mathrm{i}\mu\epsilon) \exp\left[-\frac{1}{2}\mu t\epsilon + \mathrm{i}tk\epsilon\right]$$

In direct space, this becomes

$$\begin{aligned}\phi(t, r) \mapsto F(\epsilon, t, r) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}k\, \widehat{\phi}(t, k + \mathrm{i}\mu\epsilon)\, e^{\mathrm{i}k(r+t\epsilon)}\, e^{\mu t\epsilon^2/2} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}r'\, \phi(t, r)\, e^{\mu r'\epsilon - \mu t\epsilon^2/2} \underbrace{\int_{\mathbb{R}} \mathrm{d}k\, e^{\mathrm{i}k(r-r'+t\epsilon)}}_{2\pi\delta(r+t\epsilon-r')} \\ &= \phi(t, r + t\epsilon)\, e^{\mu(r+t\epsilon)\epsilon - \mu t\epsilon^2/2}\end{aligned}$$

For $\mu = 0$, **rigid shifts** $t \mapsto t$ and $r \mapsto r + t\epsilon$.

Comparison of the standard, local, **Galilei transformation** $Y_{1/2}$ with $z = n = 2$ and the generalised, **non-local**, transformation $Y_{1/2}$ for $z = n > 2$, with the initial distribution in the restricted function space, and $\mu = 0$:

$\phi(t,r)$	non-local, $n > 2$	local, $n = 2$	
t^m	t^m	t^m	$m \in \mathbb{N}$
r^k	r^k	$(r + t\epsilon)^k$	$1 \leq k \leq n - 2$
r^{n-1}	$r^{n-1} + (n-1)!\, t\epsilon$	$\left(r + t\epsilon\right)^{n-1}$	

- t^m is always invariant
- in the local case rigid transformation of r^k
- r is invariant for the non-local case
- but some of the higher 'moments' transform !

$\Longrightarrow$ looks analogous to transformation of a distribution function of coordinates, rather than a local transformation of coordinates itself !

Comparison of the standard, local, **special transformation** X_1 with $z = n = 2$ and the generalised, **non-local**, transformation X_1 for $z = n > 2$, within the restricted function space : ($m \in \mathbb{N}$, $\mu = 0$)

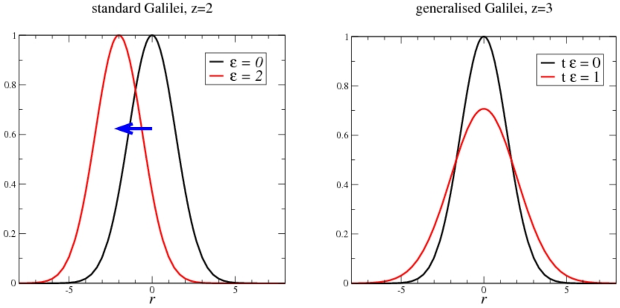
	non-local		local
ϕ	$n = 3$	$n = 4$	$n = 2$
t^m	t^m	t^m	$t^m/(1 - t\epsilon)^{m+x+\xi}$
r	r	r	$r/(1 - t\epsilon)^{1+x+\xi}$
r^2	$r^2 + 2tr\epsilon$	$r^2 + 2(x+\xi)t\epsilon$	$r/(1 - t\epsilon)^{2+x+\xi}$
	$+\left(\frac{3}{2} + x + \xi\right)t^2\epsilon^2$		
r^3		$r^3 + 6(x + \xi + 1)tr\epsilon$	$r/(1 - t\epsilon)^{3+x+\xi}$

- rigid transformations of t^m and r^k in the local case
- all moments t^m invariant in the non-local case
- r invariant in the non-local case
- but the moment r^{n-1} does transform !

Illustration for $Y_{1/2}$ in the case $n = 3$

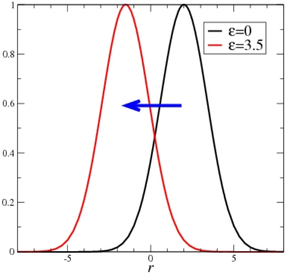
$$F(\epsilon, t, r) = \frac{1}{\sqrt{4\pi t\epsilon}} \int_{\mathbb{R}} \mathrm{d}r' \, \phi(t, r') \times \exp\left[-\frac{1}{4t\epsilon} \left(\left(r - r' - t\mu\epsilon^2 \right)^2 - 4\mu t r' \epsilon^2 - \frac{4}{3}\mu^2 t^2 \epsilon^4 \right) \right]$$

compare standard/generalised Galilei-transformation of a gaussian :

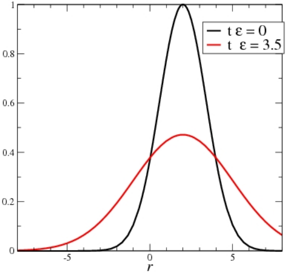


similarly, compare the transformation of a shifted gaussian :

standard Galilei, shifted, z=2



generalised Galilei, shifted, z=3



- local case : rigid shift, form unchanged
- non-local case : width increases, centre unchanged

similar expressions are known for $z = 4$

Analogous integral representations can be derived for the **finite special transformation** X_1 (with $\mu = 0$) :

$$F(\epsilon, t, r) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}k \mathrm{d}r' \, e^{\mathrm{i}k(r-r')} \left(1 + \frac{tk\epsilon}{2\mathrm{i}}\right)^{2(1-x-\xi)} \times \phi\left(t\left(1+\frac{tk\epsilon}{2\mathrm{i}}\right)^{-3}, r'\left(1+\frac{tk\epsilon}{2\mathrm{i}}\right)^{-2}\right)$$

$$\text{if } z = n = 3$$

$$F(\epsilon, t, r) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}k \mathrm{d}r' \, e^{\mathrm{i}k(r-r')-(x+\xi-2)\epsilon t k^2} \, \phi\left(e^{-2\epsilon t k^2} t, e^{-\epsilon t k^2} r'\right)$$

$$\text{if } z = n = 4$$

B.3 Co-variant two-point functions

$$F = F(t_1, t_2; r_1, r_2) = \langle \phi_1(t_1, r_1)\phi_2(t_2, r_2)\rangle$$

Distinguish the cases (i) n even and (ii) n odd. Set $XF = 0$.

(1) n even. Variables : $u := t_1 - t_2$, $v := t_1/t_2$, $r := r_1 - r_2$

$$\begin{aligned} F &= F(u, v, r) = t_2^{-(x_1+x_2)/n} \, f\left(ru^{-1/n}\right) \\ &\times \quad (v-1)^{-\frac{2}{n}[(x_1+x_2)/2+\xi_1+\xi_2-n+2]} \, v^{-\frac{1}{n}[x_2-x_1+2\xi_2-n+2]} \end{aligned}$$

(2) n odd. Variables : $u := t_1 + t_2$, $v := t_1/t_2$, $r := r_1 - r_2$.

$$\begin{aligned} F &= F(u, v, r) = t_2^{-(x_1+x_2)/n} \, f\left(ru^{-1/n}\right) \\ &\times \quad (v+1)^{-\frac{2}{n}[(x_1+x_2)/2+\xi_1+\xi_2-n+2]} \, v^{-\frac{2}{n}[x_2-x_1+\xi_1-\xi_2]} \end{aligned}$$

In **both cases**, the last scaling function f is given by :

$$f^{(n-1)}(y) + \mu_1 y f(y) = 0$$

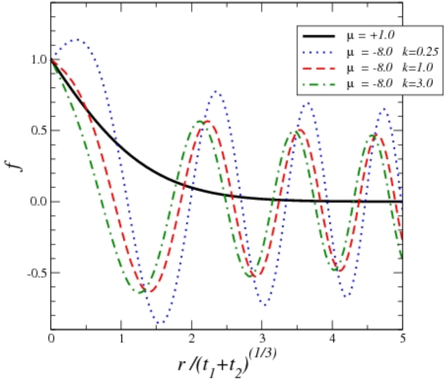
general solution in terms of hypergeometric functions ${}_0F_{n-2}$.

Illustration for the case $z = n = 3$:

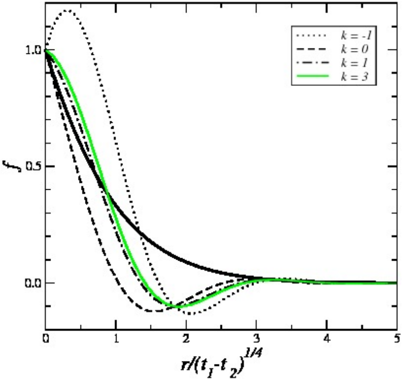
$$f(y) = f_1 \mathrm{Ai}\left(-\mu_1^{1/3} y\right) \quad ; \quad \mu_1 > 0$$

$$f(y) = f_1 \mathrm{Ai}\left(|\mu_1|^{1/3} y\right) + f_1 k \mathrm{Bi}\left(|\mu_1|^{1/3} y\right) \quad ; \quad \mu_1 < 0$$

sign of μ_1 might be used to distinguish between non-conserved and conserved dynamics



f independent of scaling dimensions $\Longrightarrow$ **super-universality** similar results are found in the case $z = n = 4$:



f independent of scaling dimensions $\Longrightarrow$ **super-universality**

Tests of LSI for $z \neq 2$:

- spherical model with conserved order-parameter, $T = T_c$, $z = 4$ — BAUMANN & MH 06
- Mullins-Herring model for surface growth, $z = 4$ — RÖTHLEIN, BAUMANN, PLEIMLING 06
- spherical model with long-ranged interactions, $T \leq T_c$, $0 < z = \sigma < 2$ — CANNAS ET AL. 01 ; BAUMANN, DUTTA, MH 07 ; DUTTA 08
- ferromagnets at their critical point (Ising, XY), $z \approx 2.0 - 2.2$ — MH, ENSS, PLEIMLING 06 ; ABRIET & KAREVSKI 04
- critical particle-reaction models (DP ?, NEKIM, Voter-Potts-3), $z \approx 1.6 - 2$ — ÓDOR 06 ; CHATELAIN, DE OLIVEIRA, TOMÉ 11
- particle-reaction models with Lévy-flight transport, $0 < z = \eta < 2$ — DURANG & MH 09

important : consideration of invariant differential equation

NB : **all** of the exactly solved models in this list are **markovian** !

Conclusions & Outlook

topics not discused here :

- calculation of two-time correlators
- extend $\mathfrak{sch}(d)$ to $\mathfrak{conf}(d+2)$
- new algebras (conformal galiléen, exotic conformal galiléen)
- relationship with string theory – AdS/CFT correspondence
- $\mathfrak{age}(d)$, $\mathfrak{sch}(d)$ have ∞-dimensional extensions
- how to generalise towards arbitrary values of $z \neq 2$
- non-local representations and fractional derivatives for $z \neq 2$
- logarithmic ageing/Schrödinger invariance

unsolved open questions :

- justify hypothesis of Galilei-invariance
- locality problems (global persistence & Markov property)
- prove LSI for non-linear equations
- how to treat LSI in master equations ?