Holographic RG: Flow diagrams, Fermions and Effective Lagrangians

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School and Workshop on Applied String Theory
Esfahan; May 3-7, 2011

References

I. Heemskerk and J. Polchinski (arXiv:1010.1264)
T. Faulkner, H. Liu and M. Rangamani (arXiv:1010.4036)
D. Nickel and D. T .Son (arXiv:1009.3094)
T. Faulkner and J. Polchinski (arXiv:1001.5049)
S. Sachdev (arXiv:1012.0299)
Daniel Elander, GM, Hiroshi Isono (1104.xxxx)
Kachru, Liu and Mulligan
Avinash Dhar, GM, Spenta Wadia

AdS/CFT

- Original AdS/CFT duality relates a given CFT (e.g. $\mathcal{N} = 4$ SYM) to a given geometry ($AdS_5 \times S^5$).
- Since it also maps deformations of the geometry to operators in the CFT, we can map other geometries to deformed CFT's.
- Interesting relevant deformations of CFT's can end up at new fixed points, which can indicate new phases.
- This corresponds to new AdS regions in the deformed geometry.

Geometry/FT

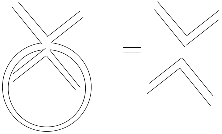
- The RG flows away from the fixed points are also interesting to study, although they are typically non-universal and would depend on the renormalization scheme.
- A precise formulation of “integrating out degrees of freedom” a la AdS/CFT can let us explore the space of field theories. Heemskerk + Polchinski, Faulker+ Liu + Rangamani
- Holographic RG gives us a natural framework to discuss phenomenological Lagrangians, e.g. those of Faulkner + Polchinski, Nickel + Son, Sachdev, ...

A matrix field theory

Consider a 3D field theory of $N \times N$ matrices:

$$S = \int d^3x \, [\mathrm{Tr} \left\{ (\partial_i M)^2 + \frac{\lambda}{N^2} M^6 \right\} + \frac{g\Lambda}{N^2} (\mathrm{Tr} M^2)^2]$$

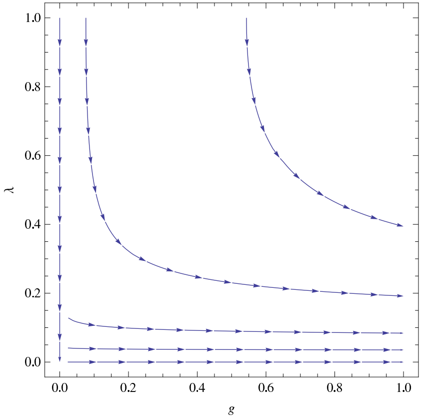
The coupling λ is marginally irrelevant.
The double trace operator gets generated through a diagram like



These are summarized by the beta-functions

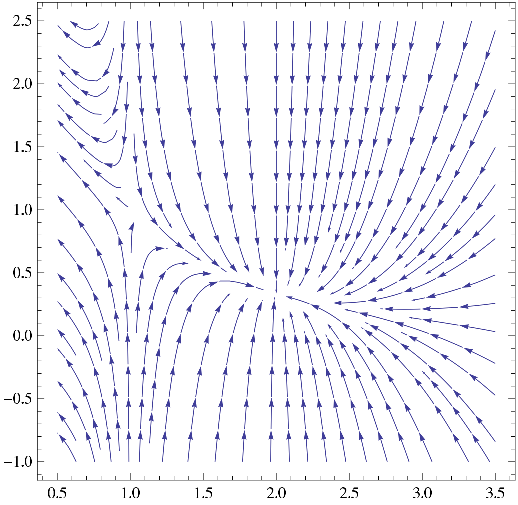
$$\dot{\lambda} = -\Lambda \frac{\partial}{\partial \Lambda} \lambda = -a\lambda^2, \dot{g} = g - b\lambda g, \quad a, b > 0$$

RG flow in the field theory



The UV fixed point at the origin is the trivial fixed point.
Question: is there a strongly coupled IR fixed point? cf. Wilson Fisher f.p. for ferromagnetic phase transitions.

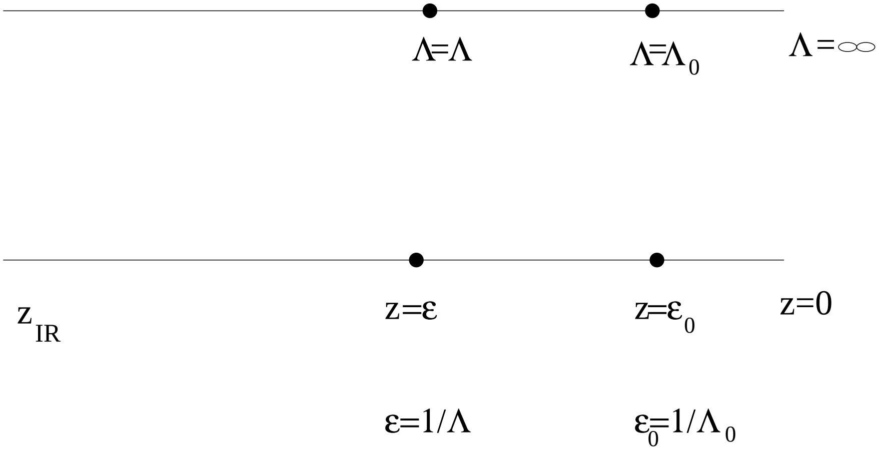
RG flow in gravity



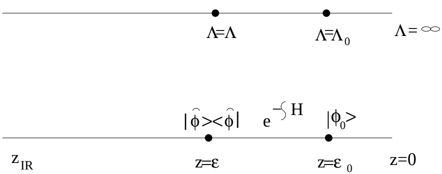
Plan

- Review of Wilsonian RG in AdS/CFT
- Flow diagrams
- Holographic RG for fermions
- Phenomenological Lagrangians
- Conclusions

Cut-off field theory= Gravity with a finite boundary



Wilsonian RG in AdS/CFT



AdS/CFT:

$$Z_{CFT}(\Lambda_0,\phi_0) \equiv \int DM|_{\Lambda_0}\ \exp[S_0^d[M] + \int d^dx\,\phi_0(x)\mathcal{O}(x)] =$$

$$Z_{bulk}(\epsilon_0,\phi_0) = \int D\phi|_{\phi(x,\epsilon_0)=\phi_0(x)} \exp\left(S^{d+1}[\phi]\right)$$

$$= \langle IR|P\ e^{\int_{\epsilon_0}^{z_{IR}} dz\ H}|\phi_0(x)\rangle$$

$$= \int D\tilde{\phi}(x)\langle IR|Pe^{-\int_{\epsilon}^{z_{IR}} dz\ H}|\tilde{\phi}(x)\rangle\langle\tilde{\phi}(x)|Pe^{-\int_{\epsilon_0}^{\epsilon} dz\ H}|\phi_0(x)\rangle$$

The Dirichlet B.C. corresponds to the “standard quantization”.

Wilsonian RG in AdS/CFT

$$
\begin{aligned}
&Z(\epsilon_0, \phi_0) \\
&= \int D\tilde{\phi}(x) \int D\phi|_{\phi(\epsilon,x)=\tilde{\phi}(x)} e^{S^{d+1}[\phi]} \langle \tilde{\phi}(x)| P e^{-\int_{\epsilon_0}^{\epsilon} dz\ H} |\phi_0(x)\rangle \\
&= \int D\tilde{\phi}(x) \int DM|_\Lambda e^{S_0^d[M] + \int d^d x \tilde{\phi}(x)\mathcal{O}(x)} \langle \tilde{\phi}(x)| P e^{-\int_{\epsilon_0}^{\epsilon} dz\ H} |\phi_0(x)\rangle \\
&\equiv \int DM|_{\Lambda_0} \ \exp[\tilde{S}_0^d[M] + \int d^d x \tilde{\phi}_0(x)\mathcal{O}(x)]
\end{aligned}
$$

Integrating out the “fast variables” amounts to computing the wave function.

Example of a scalar probe

- In principle, the above works when we consider all possible couplings $\phi_\alpha \mathcal{O}_\alpha$ involving a complete set of local operators $\mathcal{O}_\alpha$, including multi-traces.
- However, let us assume, for the moment, that we can treat one of the bulk fields ϕ in a "probe approximation" without back-reacting on the others. If $\phi(z, x)$ is a scalar field, then

$$S_0^{d+1}[\phi] = \int dz d^d x \, \sqrt{g} \left(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2\right)$$

If the geometry is AdS, then

$$\langle \tilde{\phi}(x) | P \, e^{-\int_{\epsilon_0}^{\epsilon} dz \; H} | \phi_0(x) \rangle = \exp[S_H],$$
$$S_H = \int d^d k \sqrt{\gamma} \left(-\frac{1}{2} f(\epsilon, k) \tilde{\phi}(k) \tilde{\phi}(-k) + J(\epsilon, k) \tilde{\phi}(-k) \right) + C(\epsilon)$$
$$\epsilon \partial_\epsilon S_H = \frac{1}{2} \int \sqrt{\gamma} \left(-\left(\frac{\partial S_H}{\partial \tilde{\phi}}\right)^2 + (\epsilon^2 k^2 + m^2) \tilde{\phi}^2 \right) \qquad (1)$$

The holographic RG transformation, translated to field theory, becomes

$$\int DM|_{\Lambda_0} \exp[S_0^d[M] + \int d^d x \phi_0(x)\mathcal{O}(x)]$$

$$= \int D\tilde{\phi}(x) e^{S_H[\tilde{\phi},\epsilon;\phi_0,\epsilon_0]} \int DM|_\Lambda e^{S_0^d[M]+\int d^d x \tilde{\phi}(x)\mathcal{O}(x)}$$

$$= \int DM|_\Lambda \exp\left[\tilde{S}_0^d[M] + \int d^d k \sqrt{\gamma}\left(\frac{\mathcal{O}(k)\mathcal{O}(-k)}{2\gamma f(z,k)} + \frac{J(z,k)\mathcal{O}(-k)}{f(k)} + \ldots\right)\right]$$

If $\mathcal{O}(x)$ is a single trace operator, we see that double trace operators emerge out of the holographic RG transformation. More accurately, the bulk action contains counterterms, and the double trace coupling goes as $g(k) \sim 1/(f(k) - \Delta_-)$.

Detour: if $\tilde{\phi}$ is a “light” field, cannot integrate it, and this gives rise to an additional, emergent, dynamical variable......

Phenomenological Lagrangians: a preview

Then

$$Z(\Lambda_0, \phi_0) = \int D\tilde{\phi}(x) DM|_\Lambda \ \exp\left(S_{total}[M(x), \tilde{\phi}(x), \phi_0(x)]\right)$$

$$S_{total}[M(x), \tilde{\phi}(x), \phi_0(x)] = S_0^d[M] + S_H[\tilde{\phi}(x), \phi_0(x)] + \int d^d x \tilde{\phi}(x) \mathcal{O}(x)$$

The holographic representation of the low energy effective action has an additional dynamical variable $\tilde{\phi}(x)$. One way of interpreting it is as a random source. Alternatively, it can be interpreted as an “emergent dynamical field”. Furthermore, S_{total} involves the non-dynamical bulk field ϕ_0, which gives a coupling between the UV and the IR fields.

In a model involving bulk gauge fields $A_M(z,x)$, we have, schematically

$$S_{total}[M(x), \tilde{A}_M(x), A_{0,\mu}(x)] = S_0^d[M] + S_H[\tilde{A}_\mu(x), \varphi(x), A_{0,\mu}] + \int d^d x \tilde{A}_\mu(x)\mathcal{O}_\mu(x)$$

where $A_\mu(x)$ is an emergent $U(1)$ field coupled to the low energy matter sector $M(x)$, $A_{0,\mu}(x)$ refers to the electromagnetic $U(1)_{ext}$, and $\varphi(x)$ (derived from A_z) denotes a Goldstone boson corresponding a symmetry breaking $U(1) \times U(1)_{ext} \to U(1)$. Nickel+Son, Faulkner+Liu +Rangamani, Sachdev

Beta functions

Let us return to the Schrodinger flow of S_H.
The flow of the double trace coupling is given by

$$\epsilon\partial_\epsilon f(\epsilon, k) = df(k) - f(k)^2 + \epsilon^2 k_\mu k_\mu + m^2 R^2$$

If we identify $\epsilon = 1/\Lambda$ (which is true for sufficiently small ϵ Susskind+Witten), then this looks like

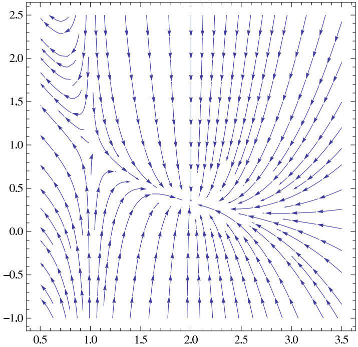
$$\beta_{f(k)} = -\Lambda \frac{d}{d\Lambda} f(z, k) = df(k) - f(k)^2 + \frac{k_\mu k_\mu}{\Lambda^2} + m^2 R^2$$

The appearance of explicit cut-off factors makes such an equation difficult to interpret; in particular, locations of fixed points would appear to depend on Λ! It is not difficult to get around this difficulty, by going to "dimensionless momenta" $\bar{k}_\mu = \epsilon k_\mu$, in terms of which the beta-function equations become Elander+Isono+GM

$$\beta_{f(\bar{k})} = (d - \bar{k}_\mu \frac{d}{d\bar{k}_\mu}) f(\bar{k}) - f(\bar{k})^2 + \bar{k}_\mu \bar{k}_\mu + m^2$$

The RHS is now independent of the cut-off, at the expense of introducing coupling between various momentum modes of $f(\bar{k})$. Define $f(\epsilon, \bar{k}) \equiv \sum_n f_n(\epsilon)(\bar{k}_\mu \bar{k}_\mu)^2$.

$$\dot{f}_0 = df_0 - f_0^2 + m^2$$
$$\dot{f}_1 = (d-2)f_1 - 2f_0 f_1 + 1, \ \ldots$$



- The two fixed points are clearly visible. Their locations on the f_0-axis are
  $IR : f_0 = \Delta_+ = d/2 + \nu$,
  $UV : f_0 = \Delta_- = d/2 - \nu$.
  Here $\nu = \sqrt{d^2/4 + m^2 R^2}$.
- The dimension of $\mathcal{O}$ at the IR fixed point turns out to be $\Delta_{\mathcal{O}} = \Delta_+$: this fixed point corresponds to the "standard quantization".
- The dimension of $\mathcal{O}$ at the UV fixed point is $\Delta_{\mathcal{O}} = \Delta_-$, hence this fixed point corresponds to the "alternative quantization". The AdS functional integral is specified by a Neumann boundary condition in this case.

- We have chosen here $m^2R^2 = -2$, which is within the window $-9/4 \le m^2R^2 \le -5/4$ where both CFT's are sensible. For this value

$$\Delta_- = 3/2 - \sqrt{9/4 - 2} = 1$$

  which matches the free field (UV) fixed point mentioned above ($\mathcal{O} = \mathrm{Tr} M^2$ has dimension $\Delta = 1$ at $\lambda = g = 0$). The above consideration suggests that there is another (IR) fixed point at strong coupling $g_* = \infty$ where $\Delta = \Delta_+ = 2$.
- If we choose $m^2R^2 > -5/4$, formally there are still two fixed points. However, the UV fixed point in this case does not define a sensible CFT (the states become non-normalizable). From the viewpoint of RG, we find an infinite number of relevant operators.

Scalars in an extremal BH background

$$ds^2 = z^{-2}\left(dt^2 H(z) + dx_i^2 + dz^2/H(z)\right)$$
$$H = 1 + 3(\frac{z}{z_*})^4 - 4(\frac{z}{z_*})^3 \quad (2)$$

The RG flow equation becomes

$$\dot{f}(\bar{k},\epsilon) = (d - \bar{k}_\mu \frac{\partial}{\partial \bar{k}_\mu} - \epsilon\partial_\epsilon H/H)f(\bar{k},\epsilon) +$$
$$+ \frac{1}{\sqrt{H}}\left(-f^2 + \bar{k}_i\bar{k}_i + \bar{w}^2/H + m^2\right)$$

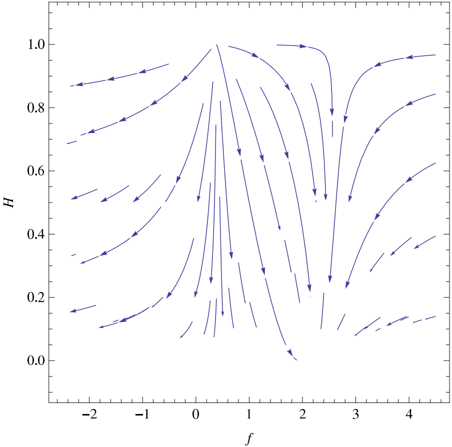
The presence of $H(\epsilon)$ introduces, again, explicit factors of ϵ. However, we should regard $H(\epsilon)$ as an additional coupling (we can replace ϵ by H!), Recall $H(\epsilon) = g_{tt}/g_{ii}$, where $g_{\mu\nu}$ couples to $T_{\mu\nu}$. The ϵ-dependence of $H(\epsilon)$ can be written as a beta-function equation by eliminating ϵ between $\epsilon\partial_\epsilon H$ and $H(\epsilon)$ (alternatively, see Kachru+Liu+Mulligan, etc.). Thus

$$\dot{H} = \beta_H(H), \dot{f}(k) = \beta_{f(k)}(f, H)$$

The flow of H is not affected by f since we are ignoring back reactions here. There is a “large” matter sector with $T_{\mu\nu} \sim O(N^2)$, and a “small” matter sector which couples to $f(k), J(k), \ldots$.

To solve these explicitly, we employ a power series $f(\bar{k}_i, \bar{w}) = \sum_{n,m} f_{n,m} (\bar{k}_i \bar{k}_i)^n \bar{w}^{2m}$.

RG flow in the $f_{0,0}$-H plane. The four fixed points are at
AdS$_4$ limit ($H = 1$): $(\Delta_-, 1)$, $(\Delta_+, 1)$,
BH horizon ($H = 0$):
$(\sqrt{3/2} - \sqrt{3/2 + m^2}, 0), (\sqrt{3/2} + \sqrt{3/2 + m^2}, 0)$.
In the diagram, $m^2 = -1$. Hence the fixed points are at
$(0.38, 1), (2.62, 1), (0.52, 0), (1.92, 0)$.



Fermions

We now consider a Dirac field $\psi(z,x)$ in the bulk

$$S_0^{d+1} = \int dz\, d^d x\, \sqrt{g}\, \left(\overline{\psi}\Gamma^M D_M \psi - m\overline{\psi}\psi\right)$$

We will again use the “probe” approximation and ignore possible back-reactions on other bulk fields e.g. the metric.

We need to specify the bulk functional integral with appropriate B.C. for various components of the fermion field. ψ is both position and momentum!

Notation: $(d+1)$-dimensional γ-matrices: $d =$ odd case

$$\Gamma^{\hat{z}} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma^{\hat{\mu}} := \begin{pmatrix} 0 & \gamma^{\hat{\mu}} \\ \gamma^{\hat{\mu}} & 0 \end{pmatrix},$$

$$\psi_{\pm} := \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \overline{\psi}_{\pm} := (\overline{\psi}_+, \overline{\psi}_-),$$

The wavefunction

Generically, ψ_- and $\overline{\psi}_+$ are the non-normalizable modes. The "standard quantization" is defined by putting "Dirichlet boundary condition" for these components. This implies choosing as the initial state the eigenstates of these fermions, which are the coherent states $|\overline{\chi}_+, \chi_-\rangle$.
We define

$$\langle \eta_+, \overline{\eta_-} | P e^{-\int_{\epsilon_0}^{\epsilon} H} |\overline{\chi}_+, \chi_-\rangle \equiv \exp[S_H(\eta, \epsilon; \chi, \epsilon_0)]$$

By evaluation, Elander + Isono+ GM

$$S_H = -\frac{1}{\kappa^2} \int \frac{d^d k}{(2\pi)^d} \left[\overline{\eta}_- F_s \eta_+ + \overline{\eta}_- S_{--} \chi_- + \overline{\chi}_+ S_{++} \eta_+ + \overline{\chi}_+ C_s \chi_-\right]$$

where

RG flows

$$\sqrt{g^{zz}}\partial_\epsilon F_s = F_s(i\gamma^\mu K_\mu)F_s + i\gamma^\mu K_\mu - 2mF_s\,,$$

$$\sqrt{g^{zz}}\partial_\epsilon S_{--} = F_s(i\gamma^\mu K_\mu)S_{--} - m_- S_{--}\,,$$

$$\sqrt{g^{zz}}\partial_\epsilon S_{++} = S_{++}(i\gamma^\mu K_\mu)F_s - m_+ S_{++}\,,$$

$$\sqrt{g^{zz}}\partial_\epsilon(\overline{\chi}_+ C_s \chi_-) = \overline{\chi}_+ S_{++}(i\gamma^\mu K_\mu)S_{--}\chi_- + \frac{\kappa^2}{(2\pi)^d}(i\gamma^\mu K_\mu F_s)\delta^d(k)\,,$$

Here, $K_\mu = k_\mu - qA_\mu$. Initial conditions for $F_s, S_{\pm\pm}, C_s$ are obtained from $\langle\eta_+, \overline{\eta}_- | \overline{\chi}_+, \chi_-\rangle$,

$$F_s = C_s = 0\,, \quad S_{++} = S_{--} = 1, \quad \epsilon = \epsilon_0\,.$$

We have solved the above RG equations (i) explicitly for the AdS geometry and (ii) in terms of Dirac wavefunctions in an arbitrary geometry.

RG solution in terms of Dirac solutions

In order to solve the flow equations, we now introduce a spinor $(M_\pm, \overline{M}_\pm)$ that satisfies the classical Dirac equations,

$$(\partial_\epsilon \mp \sqrt{g_{zz}}m)M_\pm \pm i\sqrt{g_{zz}}\,\gamma^\mu K_\mu M_\mp = 0\,,$$
$$(\partial_\epsilon \pm \sqrt{g_{zz}}m)\overline{M}_\pm \mp i\sqrt{g_{zz}}\,\overline{M}_\mp \gamma^\mu K_\mu = 0\,.$$

In terms of this spinor, we can write down general solutions to the flow equations,

$$F = M_-(M_+)^{-1} = (\overline{M}_-)^{-1}\overline{M}_+\,, \quad J_- = (\overline{M}_-)^{-1} j_-\,, \quad \bar{J}_+ = \bar{j}_+(M_+)^{-1}\,,$$

where $j_-, \bar{j}_+$ are spinors independent of ϵ. Note that the indices $\pm$ of $j, \bar{j}$ do not always reflect the actual $\Gamma^{\hat{z}}$-chirality and that depends on whether $M_+, \overline{M}_-$ contain γ-matrices or not.

Flow diagrams: AdS [standard quantization]

Define $F_s = i\gamma^{\hat{\mu}} k_{\hat{\mu}} \sqrt{g^{tt}} a$. The RG equation becomes

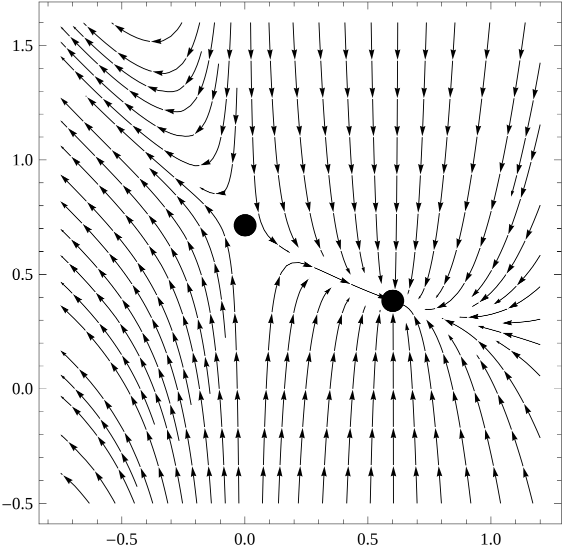
$$\epsilon\partial_\epsilon a = 1 - (2m+1)a - \epsilon^2 k_\mu k_\mu a^2$$

As in the case of the bosons, this can be interpreted as a beta-function, provided a is viewed as a function of ϵ and $\bar{k}_\mu = \epsilon k_\mu$. Writing

$$a(\bar{k}_\mu, \epsilon) = \sum_{n=0}^{\infty} a_n(\epsilon) \left(\bar{k}_\mu \bar{k}_\mu\right)^{n-1},$$

we get

$$\epsilon\partial_\epsilon a_0 = (1-2m)a_0 - a_0^2,$$
$$\epsilon\partial_\epsilon a_1 = 1 - (2m+1)a_1 - 2a_0 a_1, \cdots$$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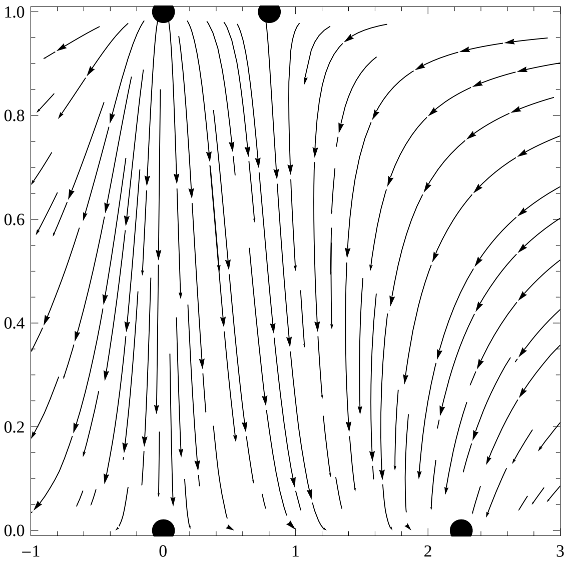
0.5
1.0

Fermionic RG flow in charged BH



Phenomenological Lagrangian

Recall

$$S_H = -\frac{1}{\kappa^2}\int \frac{d^dk}{(2\pi)^d}\left[\overline{\eta}_- F_s \eta_+ + \overline{\eta}_- S_{--}\chi_- + \overline{\chi}_+ S_{++}\eta_+ + \overline{\chi}_+ C_s \chi_-\right]$$

As discussed in the bosonic case, the full partition function of the theory (in the sector described by the fermion coupling) will be given by

$$Z(\Lambda_0,\chi) = \int D\eta(x) DM|_\Lambda \exp\left(S_{total}[M(x),\eta(x),\chi(x)]\right)$$

$$S_{total}[M(x),\eta(x),\chi(x)] = \tilde{S}_0^d[M] + S_H[\eta(x),\chi(x)] + \int d^dx\,\eta(x)\mathcal{O}_F(x)$$

Zero modes

Using the parameterization of the RG equation in terms of Dirac solutions (written as $a\times$ non-normalizable $+b\times$ normalizable), we get

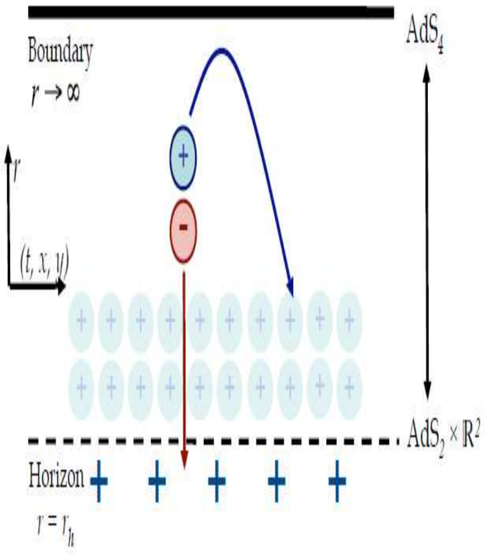
$$F_s^{-1}(q,k) = \frac{b_+(k) + O(w) + \chi_{IR}(k,w)(b_-(k) + O(w)}{a_+(k) + O(w) + \chi_{IR}(k,w)(a_-(k) + O(w))}$$
$$\chi_{IR}(k,w) \sim w^{2\nu_k},\ \nu_k = \sqrt{1/4 + m^2 + k^2 - q^2}$$

where $b_\pm(k), a_\pm(k)$ are some known functions which characterize solutions of Dirac equations. Here χ_{IR} is given by the bounrady condition in the interior.
Suppose there exists k_F such that $a_+(k_F) = 0$. That is, at such a value of k, the solution of the Dirac equation is purely normalizable. Then, for $k = k_F + \Delta k$, $a_+(k) \sim \Delta k$. Thus,

$$F_s \sim \Delta k + O(w) + O(w^{2\nu_{k_F}})$$

For $\nu_{k_F} > 1/2$, $F_s \sim \Delta k + w$.

Fermi surface



Faulkner-Polchinski

Thus, the part in S_{total} quadratic in η goes as

$$\overline{\eta}_{-}(i\partial_t + k - k_F)\eta_{+}$$

which precisely yields the semi-holographic Lagrangian of Faulkner and Polchinski (identify $M \leftrightarrow \Psi$)

$$S = S_{strong}[\Psi] + \int dtd^2k\{\eta^{\dagger}_{\vec{k}}(i\partial_t - \epsilon_{\vec{k}} + \mu)\eta_{\vec{k}} + g_{\vec{k}}\eta^{\dagger}_{\vec{k}}\Psi + g^{*}_{\vec{k}}\Psi^{\dagger}_{\vec{k}}\eta\}$$

Dyson-Schwinger for $\psi\psi^{\dagger}$:

— + — ..... — + — ..... —.....— + etc.

$$G_0(\vec{k}, w) + g_{\vec{k}}G_0(\vec{k}, w)\mathcal{G}(\vec{k}, w)G_0(\vec{k}, w) + etc.$$
$$= \frac{1}{G_0^{-1} - g_{\vec{k}}\mathcal{G}_0(\vec{k}, w0)}$$

which is the same as the connection formula obtained by solving the Dirac equation.

- The important point is to note the bilinear coupling in S_H between the dynamical (strngly coupled) IR fermion η and the non-dynamical fermion χ at the weakly coupled UV fixed point.
- Thus, the holographic RG method for fermions constitutes a derivation of effective fermion Lagrangians e.g. Faulkner+Polchinski. In particular, one can use $S_{total}[\eta, M]$ to derive the non-fermi-liquid-type dispersion relation.

Conclusions

- We reviewed the HK-FLR proposal for Wilsonian RG in AdS/CFT. This gives us a Geometry/FT duality.
- We translated the holographic RG formalism into explicit beta-functions and found RG flow diagrams with which we can locate fixed points and infer about new phases of the theory.
- We extended the holographic RG formalism to fermions.
- We derived the phenomenological Lagrangian of Faulkner and Polchinski from holographic RG.
- We discussed the example of AdS and extremal charged BH bgds for simplicity. However, our RG equations are valid for a large class of metrics. (Radial ADM, Membrane paradigm)
- From the Wilsonian viewpoint, we can have IR fixed points without UV fixed points. In an appropriate context, this might imply an AdS near-horizon geometry without an AdS geometry asymptotically. (Flat space?)