Non-relativistic Conformal Symmetries; Infinite Extensions and Logarithmic Representations

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Contents:

- Why non-relativistic conformal symmetry?
  - AdS/CFT correspondence
  - Condensed matter physics
  - Mathematical interest
  - Logarithmic Age (Henkel)
  - LGCA in the context of holography (Hosseiny+Naseh)
- Schrodinger symmetry
- Schrodinger symmetry: Infinite extension
- Galilean conformal algebra (GCA) via contraction

Contents:

- Full GCA
- Other non-relativistic conformal symmetries: *l*-Galilei algebra
- Nonrelativistic conformal algebras in 2+1 dimensions
- Representations
  - Schrodinger-Virasoro
  - GCA
- Logarithmic Conformal Field Theory (LCFT)
- Logarithmic representations:
  - Schrodinger-Virasoro algebra
  - Full GCA (LGCA)

High Tc
SC

NR
CFT

Mullah ! what
is AdS/CFT ?

QG

It is the solution
to all mankind's
problems

Fluid
Dynamics

QCD

Non-relativistic conformal symmetries

Application of AdS/CFT correspondence to Non-relativistic conformal symmetries

There exists a duality between bulk and boundary, such that the boundary theory posses Schrodinger symmetry. In other words the bulk gravity is dual to a non-relativistic CFT.

You couple a massive vector boson to gravity and extract the right metric

How do you extract a non relativistic theory out of a fundamentally relativistic theory ?

- D. T. Son, “Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry,” Phys. Rev. D 78, 046003 (2008) [arXiv:0804.3972 [hep-th]].
- K. Balasubramanian and J. McGreevy, “Gravity duals for non-relativistic CFTs,” Phys.Rev. Lett. 101, 061601 (2008) [arXiv:0804.4053 [hep-th]].

Non-relativistic conformal symmetries

Application of AdS/CFT correspondence to Non-relativistic conformal symmetries

Candidate space-time in d+2 dimensions is dual to a non-relativistic CFT in d dimesnions:

$$ds^2 = -\frac{dt^2}{z^4} + \frac{-dt\,d\eta + dx^2 + dz^2}{z^2}$$

- Space is d-1 dimensional and together with t gives rise to Schrodinger equation in d-1 dimensions

Non-relativistic conformal symmetries

Application of AdS/CFT correspondence to Non-relativistic conformal symmetries

The non-relativistic AdS/CFT correspondence implies that the asymptotic symmetry of geometries with Schrodinger isometry is an infinite dimensional algebra containing one copy of the Virasoro algebra in arbitrary dimensions.

- Asymptotic symmetry of geometries with Schrodinger isometry

Mohsen Alishahiha, Reza Fareghbal, Amir E. Mosaffa, Shahin Rouhani Phys. Lett. B675: 133-136,2009

Non-relativistic conformal symmetries

Condensed matter physics

- In this approach; strongly coupled field theories become computationally tractable and conceptually more transparent.
- Strongly correlated electron systems
- Quantum Criticality
- Critical Dynamics

….

Review: Quantum Critical Dynamics from Black Holes Sean A. Hartnoll
arXiv:0909.3553 Subjects: Strongly Correlated Electrons

M. Henkel, "Schrodinger Invariance in Strongly Anisotropic Critical Systems," J. Statist. Phys. 75 (1994) 1023 [arXiv:hep-th/9310081].

Non-relativistic conformal symmetries
Mathematical interest

- There has existed a long time interest in classification of affine symmetries independent of AdS/CFT ‘s
- Some of the names involved are:
  - Duval, Henkel, Horvathy, Fushchych, Cherniha ….

Logarithmic GCA in the context of holography

Topologically massive gravity at the critical point corresponds to logarithmic (NR) CFT

$$S = \frac{1}{16\pi G}\int dx^3 \sqrt{-g}\left[R + \frac{2}{l^2} + \frac{1}{\mu}\mathcal{L}_{cs}\right]$$

$$\mu l \to 1$$

Hosseiny + Naseh

Logarithmic Age

- In non-equilibrium situations where you do not have
  time translation invariance; you obtain a different
  algebra: Age
- We now have two distinct scaling dimensions
- Deriving the logarithmic correlators in this case is more involved.
- Fits directed percolation away from equilibrium

Malte Henkel, On logarithmic extensions of local Scale invariance, arXiv:1009.4139v1

Logarithmic Age

- What is the bulk space-time metric which has Age as its boundary symmetry ?

Schrodinger symmetry

What is the complete set of symmetries of the Schrodinger equation:

$$\left(i\partial_t + \frac{1}{2m}\partial_i\partial_i\right)\psi = 0$$

It consists of Galilean transformation symmetry:

$$H = -\partial_t \qquad P_i = \partial_i$$

$$J_{ij} = -(x_i\partial_j - x_j\partial_i) \qquad B_i = t\partial_i$$

Schrodinger symmetry

Special Schrodinger transformation

$$K = -(t x_i \partial_i + t^2 \partial_t)$$

which produces

$$x_i \to \frac{x_i}{(1+\mu t)} \qquad t \to \frac{t}{(1+\mu t)}$$

Dilation

$$D = -(2t\partial_t + x_i \partial_i)$$

which scales space and time anisotropically

$$x_i \to \lambda\, x_i \qquad t \to \lambda^2\, t$$

Schrodinger symmetry

These operators together produce the following coordinate transformations:

$$\vec{r} \rightarrow \overrightarrow{r}' = \frac{\mathcal{R}\vec{r} + \vec{V}t + \vec{a}}{\gamma t + \delta} \qquad t \rightarrow t' = \frac{\alpha t + \beta}{r t + \delta}$$

$$\alpha\delta - \beta\gamma = 1$$

Algebra admits a central charge which is mass

$$[B_i, P_j] = \mathcal{M}\delta_{ij}$$

Schrodinger symmetry: *infinite extension*

There exists a Virasoro like infinite extension to the Schrodinger algebra called Schrodinger-Virasoro:

$$T^n = -t^{n+1}\partial_t - \frac{1}{2}(n+1)t^n x_i \partial_i - \frac{1}{4}n(n+1)\mathcal{M}t^{n-1}x^2$$

$$P^m{}_i = -t^{m+\frac{1}{2}}\partial_i - (m+\frac{1}{2})t^{m-1/2}x_i\mathcal{M}$$

$$M^n = -\mathcal{M}t^n \qquad m+\frac{1}{2}, n \in \mathbb{Z}$$

M. Henkel, "Schrodinger Invariance in Strongly Anisotropic Critical Systems," J. Statist. Phys. 75 (1994) 1023 [arXiv:hep-th/9310081].

Schrodinger symmetry: *infinite extension*

Schrodinger algebra sits at the middle of this affine extension

$T^{-1} = H$ $\quad$ $T^{0} = D/2$ $\quad$ $T^{1} = K$

$P_i^{-\frac{1}{2}} = P_i$ $\quad$ $P_i^{\frac{1}{2}} = B_i$

$M^{0} = -\mathcal{M}$

Schrodinger symmetry:
infinite extension

Some of the commutators are:

$$[T^n, T^m] = (n-m)T^{n+m} \qquad [T^n, P_i^m] = (\frac{n}{2} - m)P_i^{n+m}$$

$$[T^n, M^m] = -mM^{n+m} \qquad [P_i^n, P_j^m] = (n-m)\delta_{ij}M^{n+m}$$

$$[P_i^n, M^m] = [M^n, M^m] = 0$$

Commutation relations of above operators with T^0 and M^0 helps to find representations of this algebra.

Galilean Conformal Algebra (GCA)

This algebra is obtained via contracting relativistic conformal algebra.

We let: $c \to \infty$

$$x_i \to \frac{1}{c} x_i \qquad\qquad t \to t$$

From a physical point of view, we investigate the behavior of the model in low speeds or energy.
In other words we investigate nonrelativistic limit of the model.

Galilean Conformal Algebra (GCA)

Contracting Poincare symmetry we end up with Galilean symmetry and contracting conformal symmetry we expect to end up with GCA which is conformal Galilean symmetry

Galilean Conformal Algebra (GCA)

Some familiar operators are recovered:

$$P_0 \rightarrow P_0$$

$$J_{0i} = tc\partial_i - \frac{1}{c} x_i \partial_t$$

$$\frac{1}{c} J_{oi} \rightarrow B_i = t\partial_i$$

Galilean Conformal Algebra (GCA)

We end up with GCA :

$$P_i = \partial_i \qquad H = -\partial_t$$

$$B_i = t\partial_i \qquad J_{ij} = -(x_i\partial_j - x_j\partial_i)$$

$$D = -t\partial_t - x_i\partial_i \qquad K_i = t^2\partial_i$$

$$K = K_0 = -(2tx_i\partial_i + t^2\partial_t)$$

Note that scaling operator D, scales space and time isotropically in this nonrelativistic algebra

Galilean Conformal Algebra: *Infinite Extension*

Similar to Schrodinger algebra GCA does have an infinite extension which is called *Full GCA*

$$T^n = -(n+1)t^n x_i \partial_i - t^{n+1}\partial_t$$
$$M_i^n = t^{n+1}\partial_i$$
$$J_{ij}^n = -t^n(x_i\partial_j - x_j\partial_i)$$

which in 1+1 dimensions simplifies to :

$$[T^m, M_i^n] = (m-n)M_i^{m+n} \qquad [M_i^m, M_j^n] = 0$$
$$[T^m, T^n] = (m-n)T^{m+n}$$

Galilean Conformal Algebra: *Infinite Extension*

GCA sits at the middle of Full GCA

$T^{-1} = H$ $\quad$ $T^{0} = D$ $\quad$ $T^{1} = K$

$M_i^{-1} = P_i$ $\quad$ $M_i^{0} = B_i$ $\quad$ $M_i^{1} = K_i$

Similar to the Schrodinger symmetry this symmetry can be also be realized within the AdS/CFT correspondence:

A. Bagchi and R. Gopakumar, J. High Energy Phys. 07 (2009)037; e-print arXiv:hep-th/0902.1385.

Other non-relativistic conformal symmetries: *l-Galilei algebra*

We note that in the Schrodinger-Virasoro algebra the dynamic index is 2 while in GCA it is 1 whilst both symmetries are Galilean. Question: Are other dynamical indices possible while keeping the Galilean nature ? yes

We add up these operators and try to make a finite closed algebra

We end up with the class of *l_Galilei* algebra which sometimes is called *spin-l Galilei* algebra

l-Galilei algebra

We impose a few desirable properties:

1. We have a conformal symmetry in space
2. We would like Galilean causality i.e. no $f(r)\partial_t$ operator is permitted.
3. We would like global conformal transformation in time:

$$t \to t' = \frac{\alpha t + \beta}{\gamma t + \delta} \qquad \alpha\delta - \beta\gamma = 1$$

We add up these operators and try to make a finite closed algebra

We end up with the class of *l_Galilei* algebra which sometimes is called *spin-l Galilei* algebra

l-Galilei algebra

$$H = -\partial_t \qquad D = -(t\partial_t + l x_i \partial_i)$$

$$K = t^2\partial_t + 2l x_i \partial_i \qquad J_{ij} = -(x_i\partial_j - x_j\partial_i)$$

$$P_i^n = (-t)^n \partial_i \qquad n = 1..2l$$

For the sake of closure, SCT in space has been eliminated

$$[D,H] = H \qquad [D,K] = -K \qquad [H,P_i^n] = -nP_i^{n-1}$$

$$[K,H] = 2D \qquad [D,P_i^n] = (l-n)P_i^n$$

$$[J_{ij},P_k^n] = -(P_i^n\delta_{jl} - P_j^n\delta_{il}) \qquad [J_{ij},J_{kl}] = SO(d),$$

$$[K,P_i^n] = (2l-n)P_i^{n+1}$$

M. Henkel, Phys. Rev. Lett. 78, 1940 1997; e-print arXiv:cond-mat/9610174.
J. Negro, M. A. del Olmo, and A. Rodríguez-Marco, J. Math. Phys. 38, 3786 1997.

l-Galilei algebra

Due to the last commutation relation the algebra closes if :

$$l = \frac{N}{2} \qquad N \in \mathbb{N}$$

So, dynamical scaling takes special values for this class of nonrelativistic conformal algebras. Scaling operator now scales space and time as

$$t \to \lambda^z t \qquad x \to \lambda x \qquad z = \frac{1}{l}$$

l=1/2 identifies Schrodinger algebra and l= 1 identifies GCA.

Nonrelativistic conformal algebras in 2+1 dimensions

While relativistic conformal symmetry is infinite dimensional symmetry only in d=2 we have Schrodinger-Virasoro symmetry and full GCA in any d!

We notice that we can have this extension in any d in nonrelativistic symmetries since time decouples from space and we can have SL(2,R) in the time direction.

So 2+1 is a special case ! We can have a very large algebra which has an infinite extent in space direction as well.

Nonrelativistic conformal algebras in 2+1 dimensions

We first notice that we have a conformal symmetry in space

$$L_n = -z^{n+1}\partial_z \quad \bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$$

We want Galilean causality i.e. no $f(r)\partial_t$

Global conformal transformation in time

$$t \to t' = \frac{\alpha t + \beta}{r t + \delta} \quad \alpha\delta - \beta\gamma = 1$$

Nonrelativistic conformal algebras in 2+1

Consider the following operators (free index l)

$$L_m^n = -t^n z^{m+1}\partial_z$$
$$\bar{L}_m^n = -t^n \bar{z}^{m+1}\partial_{\bar{z}}$$
$$T^n = -(t^{n+1}\partial_t + l(n+1)t^n(z\partial_z + \bar{z}\partial_{\bar{z}}))$$

Commutators:

$$[L_m^k, L_n^l] = (m-n)L_{m+n}^{k+l} \qquad [\bar{L}_m^k, \bar{L}_n^l] = (m-n)\bar{L}_{m+n}^{k+l}$$

$$[L_m^k, \bar{L}_n^l] = 0 \qquad [T^m, T^n] = (m-n)T^{m+n}$$

$$[L_m^k, T^n] = (k+mln+ml)L_m^{k+n} \qquad [\bar{L}_m^k, T^n] = (k+mln+ml)\bar{L}_m^{k+n}$$

A. Hosseiny, S. Rouhani, “Affine extension of Galilean conformal algebra in 2+1 dimensions”, J. Math. Phys. 51, (2010) 052307 [hep-th/0909.1203]

Nonrelativistic conformal algebras in 2+1

This algebra admits different central charges

$$[T^m, T^n] = (m-n)T^{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

$$[L^i_m, L^j_n] = (m-n)L^{i+j}_{m+n} + mC_s\delta_{m+n}\delta_{i+j}$$

$$[L^j_m, T^i] = (j + mil + ml)L^{i+j}_m + \frac{1}{2}jC_s\delta_{m,0}\delta_{i+j}$$

Representations of this algebra has not been worked out yet

Nonrelativistic conformal algebras in 2+1

Note that the class of l-Galilei algebras in 2+1 dimensions is a subset of this class

$$K = -t^2\partial_t - 2lt(z\partial_z + \bar{z}\partial_{\bar{z}}) = T^1$$

$$\{P_i^n\} = \{t^n\partial_z, t^n\partial_{\bar{z}}\} = \{L_{-1}^n, \bar{L}_{-1}^n\}$$

$$J = i(z\partial_z - \bar{z}\partial_{\bar{z}}) = -i(L_0^0 - \bar{L}_0^0)$$

$$H = -\partial_t = T^{-1} \qquad D = -t\partial_t - l(z\partial_z + \bar{z}\partial_{\bar{z}}) = T^0$$

Representations of Schrodinger-Virasoro

To build representaitons of Schrodinger-Virasoro algebra; inspired by relativistic CFT; we assume existence of fields which are eigestates of scaling operator:

$$[T^0, \phi] = h\phi$$

Assuming a vacuum now gives rise to eigenstates of T^0

$$\phi|0\rangle = |h\rangle$$

Representations of Schrodinger-Virasoro

Since M^0 commutes with every thing each state is also labeled by eigenvalue of M0 as well:

$$M^0 |h, M\rangle = M |h, M\rangle$$

M is central charge and will not be changed by other operators. So, should we keep it in our states ? !

Representations of Schrodinger-Virasoro

Other operators now work as ladder operators:

$$[T^0,[T^n,\phi]] = (h-n)[T^n,\phi]$$

$$T^{-n}|h\rangle \rightarrow |h+n\rangle$$

$$P^{-m}|h\rangle \rightarrow |h+m\rangle$$

$$M^{-n}|h\rangle \rightarrow |h+n\rangle$$

Representations of Schrodinger-Virasoro

Now, we define the vacuum state, annihilated by

$$M^n, \; P^m, \; T^n \, |0\rangle = 0 \qquad n, m > 0$$

As in CFT Null states exist and it is interesting to note that the first null state which appears at the second level is Schrodinger equation.

$$|\chi\rangle = ((P^{-\frac{1}{2}})^2 - 2\mathcal{M}T^{-1})|h\rangle$$

Higher Null states give rise to other differential equations.

Schrodinger Symmetry: Correlation Functions

Scaling fields under infinitesimal coordinate changes transform as:

$$[T^n, \phi(x,t)] = (t^{n+1}\partial_t + \frac{n+1}{2}t^n x.\partial_x + \frac{n(n+1)}{4}\mathcal{M}t^{n-1}x^2 + (n+1)ht^n)\phi(x,t)$$

$$[P_i^m, \phi(x,t)] = (t^{m+\frac{1}{2}}\partial_i + \left(m+\frac{1}{2}\right)\mathcal{M}t^{m-\frac{1}{2}}x_i)\phi(x,t)$$

Now, define Quasi-primary fields as fields which transform only under Schrodinger subalgebra.

Schrodinger Symmetry: Correlation Functions

We impose Schrodinger symmetry via Ward identity:

$$\langle \phi_1(x_1,t_1)\phi_2^*(x_2,t_2)\rangle = a t^{-2h_1}\Theta(t_1-t_2)\delta_{h_1,h_2}\delta_{\mathcal{M}_1,\mathcal{M}_2}\exp\left(-\frac{\mathcal{M}_1 x^2}{2t}\right)$$

Three-point functions as well:

$$F = \delta_{M_1+M_2,M_3}(t_1-t_3)^{\frac{1}{2}(h_1+h_3-h_2)}$$
$$(t_2-t_3)^{-\frac{1}{2}(h_2+h_3-h_1)}(t_1-t_2)^{-\frac{1}{2}(h_1+h_2-h_2)}$$
$$exp\left(-\frac{M_1}{2}\frac{(r_1-r_3)^2}{t_1-t_3}-\frac{M_2}{2}\frac{(r_2-r_3)^2}{t_2-t_3}\right)$$
$$g\left(\frac{[(r_1-r_3)(t_2-t_3)-(r_2-r_3)(t_1-t_3)]^2}{(t_1-t_2)(t_1-t_3)(t_2-t_3)}\right)$$

Full GCA:
Representations in 1+1dimensions

One observes that T^0 and M^0 commute and representations can be simultaneous eigenstates of both:

$$T^0|\Delta,\xi\rangle = \Delta|\Delta,\xi\rangle$$

$$M^0|\Delta,\xi\rangle = \xi|\Delta,\xi\rangle$$

Full GCA: Representations in 1+1dimensions Utilizing Contraction

- We observe that full GCA can be obtained directly from contracting the conformal algebra

- While it is not necessarily the case that contraction on Representations should work but in this case we can derive the representation of the full GCA by contraction,

Full GCA:
Representations in 1+1dimensions Utilizing Contraction

Note that conformal symmetry in 2 dimensions is composed of two Virasoro algebras:

$$L^n = -z^{n+1}\partial_z$$

$$\bar{L}^n = -\bar{z}^{n+1}\partial_{\bar{z}}$$

GCA From Contraction…

Now we impose contraction limit to Virasoro operators and observe

$$x \to \frac{x}{c} \qquad t \to t \qquad c \to \infty$$

$$\begin{aligned} L^n &= -\frac{1}{2}(t + i\frac{x}{c})^{n+1}(\partial_t - ic\partial_x) \\ &= -t^{n+1}(-ic\partial_x + \partial_t + (n+1)\frac{x}{t}\partial_x + O(\frac{1}{c}) \\ \bar{L}^n &= -t^{n+1}(ic\partial_x + \partial_t + (n+1)\frac{x}{t}\partial_x + O(\frac{1}{c}) \end{aligned}$$

$$T^n = L^n + \bar{L}^n + O(\frac{1}{c})$$

$$M^n = -i\frac{L^n - \bar{L}^n}{c} + O(\frac{1}{c})$$

So, by contraction full GCA is obtained from conformal symmetry

GCA From Contraction …

Consider the usual eigensates of the scaling operator

$$L^0|h,\bar{h}\rangle = h|h,\bar{h}\rangle \qquad \bar{L}^0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle$$

In the contraction limit we have

$$T^0|h,\bar{h}\rangle = (L^0+\bar{L}^0)|h,\bar{h}\rangle = (h+\bar{h})|h,\bar{h}\rangle$$
$$M^0|h,\bar{h}\rangle = -\frac{i}{c}(L^0-\bar{L}^0)|h,\bar{h}\rangle = \frac{i}{c}(\bar{h}-h)|h,\bar{h}\rangle$$

In other words

$$T^0|\Delta,\xi\rangle = \Delta|\Delta,\xi\rangle \qquad \Delta = h+\bar{h}$$

$$M^0|\Delta,\xi\rangle = \xi|\Delta,\xi\rangle \qquad \xi = \frac{i}{c}(\bar{h}-h)$$

GCA From Contraction, correlation functions

Similarly for two point functions:

$$\langle \phi_1(x_1,t_1)\phi_2(x_2,t_2)\rangle_{GCA} = \lim_{c\to\infty} \langle \phi_1(x_1,t_1)\phi_2(x_2,t_2)\rangle_{CFT}$$

$$= lim_{c\to\infty}\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2} z_{12}{}^{-2h_1}\bar{z}_{12}{}^{-2\bar{h}_1}$$

$$= lim_{c\to\infty} A\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2}(t_{12} + i\ \frac{x_{12}}{c})^{-(\Delta+i\,c\,\xi)}$$

$$(t_{12} - i\,\frac{x_{12}}{c})^{-(\Delta-i\,c\,\xi)}$$

$$= a\delta_{\Delta_1,\Delta_2}\delta_{\xi_1,\xi_2} t_{12}{}^{-2\Delta_1} exp\,(\frac{2\xi_1 x_{12}}{t_{12}})$$

A. Bagchi, R. Gopakumar, I. Mandal and A. Miwa, “GCA in 2d,” JHEP 1008, 004 (2010) [arXiv:0912.1090 [hep-th]].

Logarithmic Conformal Field Theory

LCFT's arise out of representations which are reducible but not decomposable:

$$L^0\phi_h(z)|0\rangle = h\phi_h(z)|0\rangle$$

$$L^0\psi_h(z)|0\rangle = h\psi_h(z)|0\rangle + \phi_h(z)|0\rangle$$

Reviews : M. Flohr, *Bits and pieces in logarithmic conformal field theory, Int. J. Mod. Phys. A 18* (2003) 4497 [arXiv:hep-th/0111228].
M.R. Gaberdiel, *An algebraic approach to logarithmic conformal field theory, Int. J. Mod.* Phys. A 18 (2003) 4593 [arXiv:hep-th/0111260

Original V. Gurarie, *Logarithmic operators in conformal field theory, Nucl. Phys. B 410 (1993) 535* [arXiv:hep-th/9303160].

Logarithmic Conformal Filed Theory

Application of Ward identities to such a pair naturally leads to logarithms in their correlation functions

$$\langle \phi | \phi \rangle = 0$$

$$\langle \phi | \psi \rangle = a z^{-2h}$$

$$\langle \psi | \psi \rangle = z^{-2h} (b - 2a \, log(z))$$

Logarithmic Conformal Filed Theory

There exists an infinite series of LCFT's which we may think of as living on the boundary of the Kac table

$$c_p = 13 - 6(p + \frac{1}{p}), \quad p = 2,3,4\cdots$$

The best studied LCFT corresponds to the theory of ghosts with c=-2, p=2

Logarithmic Conformal Filed Theory

In the c=-2 model there exist two fields with conformal weight 0, which we refer to as the identity I and the pseudo Identity w

$$L_0 I = 0$$

$$L_0 \omega = I$$

But

$$\langle I \rangle = 0, \quad \langle \omega \rangle = 1$$

Logarithmic Conformal Filed Theory

In the c=-2 model there exist a primary field of weight -1/8 with a level two Null vector, thus its four point function satisfies a hyper geometric equation:

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle \approx \left[(z_1 - z_2)(z_3 - z_4)\right]^{1/4} (\eta(1-\eta))^{1/4} F(\eta)$$

Where h is the cross ratio

$$\eta(1-\eta)\frac{d^2F}{d\eta^2} + (1-2\eta)\frac{dF}{d\eta} - \frac{1}{4}F = 0$$

Logarithmic Conformal Filed Theory

The indical equation for this hypergeometric equation has a double zero and may simply be written as:

$$\theta^2 = 0$$

Which we interpret as a nilpotent variable and use this to derive the properties of the LCFT

$$L^0|h+\theta\rangle = (h+\theta)|h+\theta\rangle$$

Expansion in powers of q yields back the original expressions

Logarithmic Conformal Filed Theory

Controversial OPE for the energy momentum tensor logarithmic pair T(z), t(z) giving rise to two central charges:

$$T(z)T(0) \approx \frac{c\,I}{2z^4} + \cdots$$

$$T(z)t(0) \approx \frac{b\,\omega + c\,I}{2z^4} + \cdots$$

Ending up with the expectations:

Logarithmic Conformal Filed Theory

Generally agreed two point functions:

$$\langle T(z)T(0)\rangle \approx 0$$

$$\langle T(z)t(0)\rangle \approx \frac{b}{2z^4}$$

Which arises out of the AdS/LCFT correspondence as well

Schrodinger Symmetry: Logarithmic Representation

For this symmetry as well we ask if logarithmic representations exist ?

Construct a "super field" using the nilpotent variable θ :

$$\mathbf{\Phi}(z,\theta) = \phi(z) + \theta\psi(z)$$

$$\mathbf{\Phi}(z,\theta)|0\rangle = |h+\theta\rangle$$

$$T^0|h+\theta\rangle = (h+\theta)|h+\theta\rangle$$

Schrodinger Symmetry: Logarithmic Representation

We can impose symmetries via Ward identity on quasi-primary fields and obtain two-point functions:

$$\langle \phi_1(x_1,t_1)\phi_2^*(x_2,t_2)\rangle = 0$$

$$\langle \phi_1(x_1,t_1)\psi_2^*(x_2,t_2)\rangle = bt^{-2h_1}\delta_{\mathcal{M}_1,\mathcal{M}_2}\exp\left(-\frac{\mathcal{M}_1 x^2}{2t}\right)$$

$$\langle \psi_1(x_1,t_1)\psi_2^*(x_2,t_2)\rangle =$$

$$t^{-2h_1}\delta_{h_1,h_2}\delta_{\mathcal{M}_1,\mathcal{M}_2}\exp\left(-\frac{\mathcal{M}_1 x^2}{2t}\right)(c-2b\,log(t))$$

A. Hosseiny and S. Rouhani, "Logarithmic correlators in nonrelativistic conformal field theory" J. Math. Phys. 51, 102303 (2010);e-print arXiv:hep-th/1001.1036

Logarithmic GCA; an Algebraic Approach

Recall that scaling fields in GCA are identified by their scaling weight and rapidity

$$[T^0, \phi] = \Delta\phi \qquad [M^0, \phi] = \xi\phi$$

Under infinitesimal changes , primary fields are transformed as:

$$[T^n, \phi] = (-1)^n$$
$$[(n+1)t^n x_i \partial_i + t^{n+1}\partial_t + (n+1)(t^h \Delta - n t^{n-1} x\xi)]\phi$$

$$[M^n, \phi] = (-1)^n[-t^{n+1}\partial_i + (n+1)t^n \xi]\phi$$

Logarithmic GCA; an Algebraic Approach

As far as we are concerned with quasi-primary fields it is easy to check that a Jordan form is possible. We can utilize nilpotent variables and observe:

$$\mathbf{\Phi} = \psi + \theta\phi \qquad \tilde{\Delta} = \Delta + \acute{\Delta}\theta \qquad \tilde{\xi} = \xi + \acute{\xi}\theta$$

$$[T^0, \Phi] = \tilde{\Delta}\Phi \qquad [M^0, \Phi] = \tilde{\xi}\Phi$$

Logarithmic GCA; an Algebraic Approach

Now, we can impose GCA symmetry on quasi-primary fields and calculate two-point functions 0f "superfields"

$$< \Phi(x_1,t_1,\theta_1)\Phi(x_2,t_2,\theta_2) >= e^{-\frac{\left(2\xi_1+\xi'(\theta_1+\theta_2)\right)x}{t}}$$

$$[at^{-2\Delta_1}(\theta_1+\theta_2) - 2a\Delta' \log t \, t^{-2\Delta_1} \, \theta_1\theta_2 + bt^{-2\Delta_1}\theta_1\theta_2]$$

$$\delta_{\xi_1,\xi_2}\delta_{\Delta_1,\Delta_2}$$

Logarithmic GCA; an Algebraic Approach

Expanding in nilpotent variables, we obtain:

$$< \phi_1 \phi_2 >= 0$$

$$< \phi_1 \psi_2 >= a e^{-2\xi_1 \frac{x}{t}} t^{-2\Delta_1} \delta_{\xi_1,\xi_2} \delta_{\Delta_1,\Delta_2}$$

$$< \psi_1 \psi_2 >= e^{-2\xi_1 \frac{x}{t}} t^{-2\Delta_1} \delta_{\xi_1,\xi_2} \delta_{\Delta_1,\Delta_2} \left[-2a\Delta' \log t - 2a\xi' \frac{x}{t} + b\right]$$

Logarithmic GCA; Contraction Approach

We can equally well obtain these results by contraction:

Consider the most general logarithmic representation in which both left and right scaling weights have Jordan cell structure:

$$L^0|h,\bar{h},1\rangle = h|h,\bar{h},1\rangle + \acute{h}|h,\bar{h},0\rangle$$
$$\bar{L}^0|h,\bar{h},1\rangle = \bar{h}|h,\bar{h},1\rangle + \acute{\bar{h}}|h,\bar{h},0\rangle\,.$$

Logarithmic GCA; Contraction Approach

Now, follow through the contraction procedure:

$$T^0|\Delta,\xi,0\rangle = T^0\big|h,0,\bar{h},0\rangle = \Delta|\Delta,\xi,0\rangle$$

$$T^0|\Delta,\xi,1\rangle = T^0\big|h,\bar{h},1\rangle = h\big|h,\bar{h},1\rangle + \bar{h}\big|h,\bar{h},1\rangle + (\acute{h}+\acute{\bar{h}})\big|h,\bar{h},0\rangle$$
$$= \Delta|\Delta,\xi,1\rangle + \acute{\Delta}|\Delta,\xi,0\rangle$$

$$M^0|\Delta,\xi,1\rangle = M^0\big|h,\bar{h},1\rangle = -i\frac{h}{c}\big|h,\bar{h},1\rangle + i\frac{\bar{h}}{c}\big|h,\bar{h},0\rangle - \frac{i}{c}(\acute{h} - \acute{\bar{h}}\big|h,\bar{h},0\rangle$$
$$= \xi|\Delta,\xi,1\rangle + \acute{\xi}|\Delta,\xi,1\rangle$$

Logarithmic GCA; Contraction Approach

So, we have

$$\acute{\Delta} = \acute{h} + \acute{\bar{h}} \qquad\qquad \acute{\xi} = \frac{\acute{h} - \acute{\bar{h}}}{c}$$

Now, we can follow on and find two point function and compare them with algebraic approach

Logarithmic GCA; Contraction Approach

For $\psi\psi$ two-point function where logarithmic term appears we have:

$$< \psi_1(z_1,\bar{z}_1)\psi_2(z_2,\bar{z}_2) >=$$

$$\left(-2a\left[\acute{h}_1 \log(z) + \acute{\bar{h}}_1 \log(\bar{z})\right] + b\right) z^{-2h_1}\bar{z}^{-2\bar{h}_1}\,\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2}$$

Logarithmic GCA; Contraction Approach

If we follow contraction limit for logarithmic GCA we obtain

$$\langle \psi_1(x_1, t_1)\psi_2(x_2, t_2)\rangle_{GCA} =$$

$$\delta_{\Delta_1,\Delta_2}\delta_{\xi_1,\xi_2} t^{-2\Delta_1} exp\left(\frac{2\xi_1 x}{t}\right)(-2a\acute{\Delta} log(t) - 2a\acute{\xi}\frac{x}{t} + b)$$

These correlators are exactly the same as those obtained via the algebraic approach.

Outlook:

- Representation Theory (SV, LSV, Full GCA, LGCA, *l*-Galilei, …

  Non-Relativistic Conformal Field Theory

- Physical examples…?

  Topologically Massive Gravity

  Percolation, Abelian Sandpile Model,…

Thank you